Discrete-Time Dynamic Consensus on the Max Value*

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Abstract. In this paper we propose a novel consensus protocol for discrete-time multi-agent systems (MAS), which solves the dynamic consensus problem on the max value, the so-called dynamic max-consensus problem. In the dynamic max-consensus problem the objective of each agent is to estimate the time-varying value of the maximum instantaneous value among the reference signals associated to the agents in the network, by exploiting only local interactions. The proposed interaction protocol enables the agents to solve this problem with an a priori bounded error, without exchange of inputs information among the agents. Furthermore, the proposed protocol can be tuned by means of a tuning parameter, enabling a trade-off between convergence time and steady-state error. We also provide a preliminary characterization of the maximum relative tracking error. Numerical simulations corroborate the theoretical analysis of the convergence properties of the proposed protocol.

Keywords: stability, convergence, dynamic max-consensus, distributed estimation, multi-agent systems, time-varying reference signals, bounded tracking error

1 Introduction

Motivation. In multi-agent systems the consensus problem consists in the design of a mechanism based on local interactions among agents in a network so that all their state variables of the agents converge or "agree" about the same value. In the consensus problem agreement is usually performed on a set of initial states associated to the agents, while in the dynamic consensus problem to each agent is associated a time-varying reference signal and the objective of the consensus problem is to make the state variables of the agents agree upon a

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function of the time-varying reference signals, such as average, median, maximum and so on.

While the literature has focused significantly on the dynamic average-consensus problem [1–9], estimating the average is not always the goal. In fact, in this paper we focus on estimating the maximum among the reference signals, i.e., we address the dynamic max-consensus problem. Applications of dynamic max-consensus protocols mainly reside in the field of distributed synchronization, such as time-synchronization [10] and target tracking [11], and network parameter estimation, such as cardinality [12] and highest/lowest node degree [13].

Thus, in this paper we propose the first dynamic consensus protocol in the literature capable of estimating with bounded error the time-varying maximum value among the set of reference signals given as input to the agents.

Literature review. In the literature the so-called max-consensus problem has been thoroughly investigated. Its objective is to make the states of the agents converge to the maximum of their initial states. The most popular max-consensus protocol consists in initializing the network to a set of values and let agent update its state at each instant of time by taking the maximum value among the value of the neighbors' state and its own state [14].

The work in [15] proposes conditions to achieve max-consensus and compute convergence rate of these protocols for different communication topologies. Only little effort has been paid to analyze slightly different but much more complicated variations of this problem. In particular, convergence results have only been provided for synchronous switching topologies [16] and for probabilistic asynchronous fixed frameworks [17]. The contribution of introducing time delays in the communications is due to [18], while [19] is the first work allowing noise in the communications. Finally, the case with agents with the possibility to join or leave the network, so-called open multi-agent systems, is addressed in [20].

If a static consensus protocol is used to perform a distributed estimation upon some time-varying quantities, known or measured by the agents, the protocol requires to be re-initialized in the whole network each time the value of the function to be estimated changes. To avoid re-initialization issues of consensus protocols, dynamic consensus protocols has to be investigated. However, to the best of our knowledge, there are not existing results addressing the dynamic max-consensus problem, object of this paper.

Main contribution. The main contribution of this paper consists in the first dynamic consensus protocol that solves the dynamic consensus problem on the max value in discrete-time. We first characterize in Theorem 1 its convergence properties for constant inputs and show that the convergence is reached with bounded and tunable steady-state relative error with finite convergence time; for time-varying inputs, whose relative change is asked to be bounded over time (see Assumption 1 in Section 3), we also give a characterization in Theorem 2 of its maximum relative tracking error.

Structure of the paper. In Section 2 technical preliminaries regarding the multi-agent framework considered in the paper are presented. In Section 3 the problem statement addressed in this work is formally presented. In Section 4 the proposed dynamic max-consensus (DMC) protocol is stated and qualitatively discussed. In Section 5 the convergence properties of the DMC protocol along with its tuning conditions are theoretically characterized. In Section 6 numerical simulations to corroborate the theoretical analysis are shown. Finally, in Section 7 concluding remarks are given.

2 Preliminaries

We consider a MAS whose pattern of interactions is represented by an *undirected* graph $\mathcal{G} = (V, E)$ where $V = \{1, \ldots, n\}$ is the set of nodes, i.e., the agents, and $E \subseteq V \times V$ is the set of edges. An edge $(i, j) \in E$, with $i \neq j$, exists in the graph if there exists a communication channel between agent i and j. Since the graph is undirected, $(i, j) \in E$ if and only if $(j, i) \in E$. Nodes that can exchange information are said to be *neighbors*. A set of neighbors $\mathcal{N}_i$ is associated to each node i, defined as $\mathcal{N}_i = \{j \in V : (i, j) \in E\}$, which represents the agents in the graph which share a point-to-point communication channel with agent i.

A *path* π_{pq} between two nodes p and q in a graph is a finite sequence of m edges $e_k = (i_k, j_k) \in E$ that joins node p to node q, i.e., $i_1 = p$, $j_m = q$ and $j_k = i_{k+1}$ for $k = 1, \ldots, m-1$. An undirected graph is said to be *connected* if there exists a path π_{ij} between any pair of nodes $i, j \in V$. The *diameter* of a graph, denoted as $\delta(\mathcal{G})$, is defined as the longest among the shortest paths among any pair of nodes $i, j \in V$. For any connected undirected graph it holds $\delta(\mathcal{G}) \leq n - 1$ with n being the number of nodes.

3 Problem statement

Consider a network of agents whose topology is represented by an undirected connected graph $\mathcal{G}$. Each agent i has access to a time-varying external reference signal $u_i(k) \in \mathbb{R}$ satisfying the next assumption[1].

Assumption 1. Each unknown exogenous reference signal is strictly positive, $u_i(k) > 0$ and their relative change is bounded by a constant $\varPi \in (0, 1)$, i.e.,

$$\frac{|u_i(k+1) - u_i(k)|}{|u_i(k)|} \leq \varPi, \quad \forall i \in V,\ \forall k \geq 0. \tag{1}$$

Neighboring agents exchange information about their own state $x_i(k) \in \mathbb{R}$ and thus cooperate according to a discrete-time local interaction protocol

$$x_i(k+1) = f_i(u_i(k), x_j(k) : j \in \mathcal{N}_i). \tag{2}$$

[1] Note that by increasing the sampling frequency of the unknown reference exogenous signals their relative change in one iteration is reduced, thus for any signal with bounded relative change there exists a sampling frequency such that Assumption 1 is satisfied.

The dynamic max-consensus problem consists in steering the agents' state to the time-varying value of the maximum value of the exogeneous reference signals

$$\bar{u}(k) = \max_{i \in V} u_i(k), \tag{3}$$

with bounded relative error

$$\exists \varepsilon \geq 0, \bar{k} \geq 0 : \quad \frac{|x_i(k) - \bar{u}(k)|}{\bar{u}(k)} \leq \varepsilon \quad \forall i \in V, \, \forall k \geq \bar{k}, \tag{4}$$

regardless of the initial condition $x(0) = [x_1, \ldots, x_n]^T$.

Objective of this paper is to propose a local interaction protocol (2) for a discrete-time MAS, which solves the dynamic consensus problem formalized in (3)-(4) under Assumption 1 by abiding the limitations of the unknown network topology $\mathcal{G}$. Performance and convergence properties of the proposed protocol are theoretically characterized.

4 Proposed Dynamic Max Consensus (DMC) protocol

The proposed local interaction protocol executed by each agent to solve the dynamic consensus problem on the max value is shown next

$$x_i(k+1) = \max_{j \in \mathcal{N}_i \bigcup \{i\}} \{\alpha \cdot x_j(k), u_i(k - mod(k,T))\} \ , \tag{5}$$

where $\alpha \in (0,1)$ is a real tuning parameter and function $mod(k,T)$ denotes the *modulo* operation, i.e., it outputs the remainder after the division of the integer k by T. For sake of clarity, we also detail in Protocol 1 the proposed Dynamic Max-Consensus (DMC) protocol which makes use of the proposed local interaction protocol in (5).

Protocol 1 takes as input a tuning parameter $\alpha \in (0,1)$, a time interval $T \in \mathbb{Z}$ and an arbitrary initialization of the state variables $x_0 = [x_{10}, \ldots, x_{1n}]^T \in \mathbb{R}^n$. At each iteration all nodes gather the state values of their neighbors and update their state according to the protocol in (5). The value of the input reference signal is updated only every T discrete time steps, since the argument $k - mod(k,T)$ changes value only every T units of discrete time.

5 Convergence properties

In this section we characterize the convergence properties of Protocol 1 for constant (see Theorem 1) and time-varying (see Theorem 2) input reference signals

In the next theorem we characterize the steady state relative estimation error, i.e., the error relative to the magnitude of the maximum value among the reference signals to be estimated at the steady state (constant inputs), and the rise time, i.e, the convergence time of the protocol from an arbitrary initial condition to the steady state value.

Theorem 1 (Steady state relative error and rise time). *Consider a MAS executing Protocol 1 with tuning parameter $\alpha \in (0,1)$ and time interval $T \in \mathbb{N}$. If graph $\mathcal{G}$ is connected and if there exists a time $k^\star$ such that $mod(k^\star, T) = 0$ and such that reference signals $u_i(k)$ are constant and positive for $k \geq k^\star$ and $\forall i \in V$, then given*

$$\bar{k} = \max\left\{ \left\lceil \log_\alpha \frac{\bar{u}(k^\star)}{\max_{i \in V} x_i(k^\star)} \right\rceil, 0 \right\} + k^\star + \delta(\mathcal{G}), \tag{6}$$

for any time $k \geq \bar{k}$ and any initial condition $x(0) \in \mathbb{R}^n$ the relative error is bounded by

$$e(k) = \max_{i \in V} \frac{|x_i(k) - \bar{u}(k^\star)|}{\bar{u}(k^\star)} \leq 1 - \alpha^{\delta(\mathcal{G})}, \tag{7}$$

where $\delta(\mathcal{G})$ denotes the diameter of graph $\mathcal{G}$, $\bar{u}(k^\star)$ is defined as in (3) *and $\lceil \cdot \rceil$ is the round up function*[2]*.*

Proof. By hypothesis, there exists a time $k^\star$ after which all reference signals $u_i(k)$ are constant and positive. Thus, in the following we omit the dependence of $u_i(k)$ and $\bar{u}(k)$ from k and let $k \geq k^\star$.

Consider the worst case scenario,

$$\max_{i \in V} x_i(k^\star) \geq \bar{u}$$

and

$$\alpha \max_{i \in V} x_i(k^\star) \geq \bar{u},$$

then, according to (5), it holds

$$\max_{i \in V} x_i(k^\star + 1) \leq \alpha \max_{i \in V} x_i(k^\star),$$

[2] The round up function $\lceil \cdot \rceil$ denotes the operation of rounding the argument to the first integer greater than or equal to the argument.

Protocol 1: Dynamic Max-Consensus (DMC)

Input : Tuning parameter $\alpha \in (0,1)$;
Time interval $T \in \mathbb{Z}$;
Initial estimation $x_{i0} \in \mathbb{R}$ for any $i \in V$.
Set : $x_i(0) \leftarrow x_{i0}$ for any $i \in V$
for $k = 0, 1, 2, \ldots$ **each node** i **do**
  Gather $x_j(k)$ from each neighbor $j \in \mathcal{N}_i$
  Update the current state according to
  $x_i(k+1) = \max_{j \in \mathcal{N}_i \cup \{i\}} \{\alpha \cdot x_j(k), u_i(k - mod(k,T))\}$

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and by induction, for $k \geq k^\star$ it holds

$$\max_{i \in V} x_i(k) \leq \alpha^{k-k^\star} \max_{i \in V} x_i(k^\star).$$

Since the maximum among all agents' state is decreasing, then there exists $k_1 \geq k^\star$ such that

$$\alpha^{k_1-k^\star} \max_{i \in V} x_i(k^\star) \leq \bar{u}.$$

By simple manipulation it follows that the smallest value of k_1 can be computed by

$$k_1 = \max\left\{\left\lceil \log_\alpha \frac{\bar{u}}{\max\limits_{i \in V} x_i(k^\star)} \right\rceil, 0\right\} + k^\star . \tag{8}$$

for which it holds

$$\max_{i \in V} x_i(k_1) \leq \bar{u}.$$

Clearly, if $\max\limits_{i \in V} x_i(k^\star) \leq \bar{u}$, $k_1 = k^\star$. Now, let $V_1 = \{i \in V : u_i = \bar{u}\}$ denote the set of agents whose reference signal holds the maximum steady state value at $k = k_1$. Obviously $V_1 \subseteq V$. By protocol (5), it follows that $\forall i \in V_1$, $x_i(k) = \bar{u}$ for all $k \geq k_1$.

Let us now consider the set of one-hop neighbors of nodes in set V_1 which at time $k = k_1$ have state value less than $\bar{u}$ and denote it by V_2. Formally, $V_2 = \{i \in V : (i,j) \in E,\ j \in V_1, i \notin V_1\}$. Thus, for all $i \in V_2$, the state update rule (5) reduces to

$$x_i(k_1 + 1) = \max\{\alpha\bar{u}, u_i\},$$

because all agents $i \in V_2$ have a neighbor $j \in V_1$ with state value $x_j = \bar{u}$. Thus, it holds

$$x_i(k_1 + 1) \in (\alpha\bar{u}, \bar{u}) \ .$$

By induction, define the set

$$V_{\ell+1} = \left\{ i \in V : (i,j) \in E, j \in \bigcup_{s=1,\dots\ell} V_s, i \notin \bigcup_{s=1,\dots\ell} V_s \right\} .$$

By repeating this simple process up to $V_{\delta(\mathcal{G})} \equiv V$ where $\delta(\mathcal{G})$ is the graph diameter, i.e., the longest shortest path among any pair of nodes in the network, which exists because graph $\mathcal{G}$ is connected, for $i \in V_{\delta(\mathcal{G})}$ it holds

$$x_i(k_1 + \delta(\mathcal{G})) > \max\{\alpha^{\delta(\mathcal{G})}\bar{u}, u_i\} \geq \alpha^{\delta(\mathcal{G})}\bar{u} \, .$$

Thus, for all $k > k_1 + \delta(\mathcal{G})$ it holds $x_i(k) \in \left(\alpha^{\delta(\mathcal{G})}\bar{u}, \bar{u}\right)$ $\forall i \in V$. Therefore, it follows

$$\frac{|x_i(k) - \bar{u}|}{\bar{u}} \leq 1 - \alpha^{\delta(\mathcal{G})} \quad \forall k \geq \bar{k}, \quad \forall x(0) \in \mathbb{R}^n,$$

thus completing the proof.

From the result of Theorem 1 it follows that, according to (7), to minimize the steady state relative error we need to choose $\alpha \approx 1$, $\alpha < 1$. On the other hand, α determines the rise time or convergence time to the steady state according to (6), with smaller values of α giving a smaller rise time (since both argument and basis of the logarithm are smaller than 1). Thus, the value of α trades-off convergence time for steady-state relative estimation error.

It follows that a pragmatic design criterion for the choice of α is to first fix the desired steady-state relative error and then choose the smallest α which allows to satisfy the error performance constraint to minimize the rise time.

In the case of $\alpha = 1$ the proposed protocol becomes the standard max-consensus protocol [14]. This protocol allows to exactly solve the max-consensus problem in a finite number of steps but with the strong assumption that all initial state variables are initialized strictly lesser than the maximum reference signal in the network. Thus, for $\alpha = 1$, the protocol is not robust to re-initialization and therefore is not able to track time-varying reference signals and solve the dynamic max-consensus problem. Our choice of $\alpha \approx 1$, $\alpha < 1$ allows to avoid this issue, ensuring robustness to re-initialization and ability of tracking the time-varying max-value among inputs, while preventing to reach exact consensus for static inputs.

Parameter T of the DMC protocol does not influence the steady-state relative error, nor it influences the rise time significantly, thus its choice is arbitrary in this case. On the other hand, the choice of parameter T influences the maximum tracking error of the DMC protocol, which is characterized in Theorem 2.

Next, we characterize the maximum tracking error in the case of time-varying input reference signals under Assumption 1.

Theorem 2 (Tracking error with time-varying inputs). *Consider a MAS executing Protocol 1 with tuning parameter $\alpha \in (0, 1)$ and $T \in \mathbb{N}$. Consider time-varying input reference signals $u_i(k)$ under Assumption 1. If graph $\mathcal{G}$ is connected, and the tuning parameters α and T satisfy*

$$\alpha < 1 - \Pi, \quad T = \left\lceil \frac{\delta(\mathcal{G})}{1 - \log_\alpha(1 - \Pi)} \right\rceil , \tag{9}$$

then there exists $\bar{k} \in \mathbb{Z}$ such that for any time $k \geq \bar{k}$ and any initial condition $x(0) \in \mathbb{R}^n$ the relative error is bounded by

$$e(k) = \max_{i \in V} \frac{|x_i(k) - \bar{u}(k)|}{\bar{u}(k)} \leq \max \left\{ \frac{1}{(1 - \Pi)^T} - 1, 1 - \frac{\alpha^{\delta(\mathcal{G})}}{(1 + \Pi)^{T + \delta(\mathcal{G})}} \right\}$$

where $\delta(\mathcal{G})$ denotes the diameter of graph $\mathcal{G}$ and Π is the maximum relative change of the inputs according to (1).

Proof. Consider any set of inputs $u(0) \in \mathbb{R}^n_+$ and initial conditions $x(0) \in \mathbb{R}^n$ at time $k = 0$. Define the maximum relative error at a generic time k is

$$e(k) = \max_{i \in V} \frac{|x_i(k) - \bar{u}(k)|}{\bar{u}(k)} . \tag{10}$$

Thus, $e(0)$ can be any value in $\mathbb{R}_+$. Let us denote $\bar{x} = \max_i x_i$, $\underline{x} = \min_i x_i$, $\bar{u} = \max_i u_i$, $\underline{u} = \min_i u_i$. In each time interval $[\lambda T, (\lambda+1)T]$ with $\lambda \in \mathbb{N}$ the inputs are not updated and then can be considered constant. Two cases may occur, either

$$\bar{x}(\lambda T) > \bar{u}((\lambda-1)T), \tag{11}$$

or

$$\bar{x}(\lambda T) \leq \bar{u}((\lambda-1)T). \tag{12}$$

If (11) holds for λ, then by (5) and Assumption 1 it holds

$$\bar{x}((\lambda-1)T) = \frac{1}{\alpha^T}\bar{x}(\lambda T), \tag{13}$$

$$\bar{u}(((\lambda-2)T) \leq \frac{1}{(1-\Pi)^T}\bar{u}((\lambda-1)T). \tag{14}$$

Given (11) verified for λ we aim to prove that (11) holds also for $(\lambda-1)$, i.e.,

$$\bar{x}((\lambda-1)T) > \bar{u}((\lambda-2)T)$$

Then, we use (13) and impose that is greater than the upperbound on the input given by (14), i.e.,

$$\bar{x}(\lambda T) > \frac{\alpha^T}{(1-\Pi)^T}\bar{u}((\lambda-1)T).$$

Given α and Π satisfying (9), i.e., $\alpha < 1-\Pi$ it follows that (11) holds for $\lambda_1 - 1$ and, by induction, it holds also for any $\lambda \in [0, \lambda_1]$.

Since (11) and (12) are mutually exclusive, it is straightforward to notice that there exists λ^* such that for any $\lambda \in [0, \lambda^*]$ condition (11) holds true, while for any $\lambda > \lambda^*$ condition (12) holds true instead.

One can verify that such λ^* exists also from all examples in Section 6 while keeping in mind that for each time interval $[\lambda T, (\lambda+1)T]$ the maximum state must be checked at the end of the interval and the maximum input must be checked at the beginning of the interval.

Now, consider first the case in which $\lambda \in [1, \lambda^*]$. Let us compute the difference between errors at two consecutive values of λ, say λT and $(\lambda-1)T$. Since by combining (11) and (13) it follows

$$\bar{x}(\lambda T) = \frac{1}{\alpha^T}\bar{x}((\lambda+1)T) > \bar{u}(\lambda T), \tag{15}$$

then the maximum relative error at λT is

$$\begin{aligned} e(\lambda T) &= \max_{i \in V} \frac{|x_i(\lambda T) - \bar{u}(\lambda T)|}{\bar{u}(\lambda T)} \\ &= \frac{\bar{x}(\lambda T) - \bar{u}(\lambda T)}{\bar{u}(\lambda T)} \\ &= \frac{\bar{x}(\lambda T)}{\bar{u}(\lambda T)} - 1 \end{aligned}$$

The maximum relative error at $(\lambda - 1)T$ is

$$\begin{aligned} e((\lambda-1)T) &= \max_{i\in V} \frac{|x_i((\lambda-1)T) - \bar{u}((\lambda-1)T)|}{\bar{u}((\lambda-1)T)} \\ &= \frac{\bar{x}((\lambda-1)T) - \bar{u}((\lambda-1)T)}{\bar{u}((\lambda-1)T)} \\ &= \frac{\bar{x}((\lambda-1)T)}{\bar{u}((\lambda-1)T)} - 1 \\ &= \frac{1}{\alpha^T}\frac{\bar{x}(\lambda T)}{\bar{u}((\lambda-1)T)} - 1 \\ &\geq \left(\frac{1-\varPi}{\alpha}\right)^T \frac{\bar{x}(\lambda T)}{\bar{u}(\lambda T)} - 1, \end{aligned}$$

where in the last two steps, equations (13) and (14) are used to find the lowerbound. To ensure $e(\lambda T) < e((\lambda - 1)T)$, we ask that the exact value of $e(\lambda T)$ is lesser than the lowerbound of $e((\lambda - 1))$, as follows

$$\begin{aligned} \frac{\bar{x}(\lambda T)}{\bar{u}(\lambda T)} - 1 &< \left(\frac{1-\varPi}{\alpha}\right)^T \frac{\bar{x}(\lambda T)}{\bar{u}(\lambda T)} - 1 \\ \frac{\bar{x}(\lambda T)}{\bar{u}(\lambda T)} &< \left(\frac{1-\varPi}{\alpha}\right)^T \frac{\bar{x}(\lambda T)}{\bar{u}(\lambda T)} \\ 1 &< \left(\frac{1-\varPi}{\alpha}\right)^T \\ 1 &< \frac{1-\varPi}{\alpha} \\ \alpha &< 1 - \varPi. \end{aligned}$$

Therefore, by choosing the tuning parameter α according to (9), it is possible to ensure that the relative error contracts for each $\lambda \in [0, \lambda^*]$.

Now, consider the case in which $\lambda > \lambda^*$. In this case we cannot ensure that at λT the relative error decreases. Thus, we consider a T large enough so that the system reaches an equilibrium point at λT. The equilibrium point at λT is characterized in the proof of Theorem 1 and it is such that equation (12) holds with the equality for $\lambda \geq \lambda^* + 1$, i.e.,

$$\bar{x}(\lambda T) = \bar{u}((\lambda - 1)T), \tag{16}$$

In order to find a lower bound to T we need to find a lower bound to 6. Since the only free term is the logarithm

$$\log_\alpha \frac{\bar{u}(\lambda T)}{\bar{x}(\lambda T)},$$

to find its greatest value we must compute the minimum argument. To do so, we take the minimum value of $\bar{u}(\lambda T)$ which is $(1 - \varPi)^T \bar{u}((\lambda - 1)T)$ and the

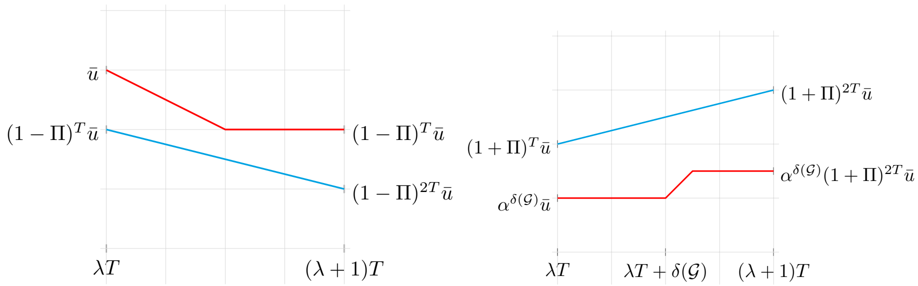

Fig. 1. Dynamics plot for Theorem 2, with $\bar{u}$ being $\bar{u} = \max_i u_i((\lambda-1)T)$. In the left figure the red curve denotes the maximum among the states while in the right figure it denotes the minimum. The blue curve always denotes the maximum among all inputs.

maximum value of $\bar{x}(\lambda T)$ which is $\bar{u}((\lambda-1)T)$ accordingly to 16, thus obtaining

$$R = \frac{\bar{u}(\lambda T)}{\bar{x}(\lambda T)} = \frac{(1-\Pi)^T \bar{u}((\lambda-1)T)}{\bar{u}((\lambda-1)T)} = (1-\Pi)^T \,, \tag{17}$$

which is constant and does not depend on the initial state $x((\lambda-1)T)$. Now, by considering the notation in Theorem 1, consider $k^\star = \lambda T$ and $T = \bar{k} - k^\star$. Thus, by 6 and 17 it follows

$$\begin{aligned} T &\geq \log_\alpha(R) + \delta(\mathcal{G}) \\ &\geq \log_\alpha(1-\Pi)^T + \delta(\mathcal{G}) \\ &\geq T\log_\alpha(1-\Pi) + \delta(\mathcal{G}) \\ &\geq \tfrac{\delta(\mathcal{G})}{1-\log_\alpha(1-\Pi)} \,, \end{aligned}$$

which is condition (9).

Having chosen T accordingly to (9), consider $\lambda \geq \lambda^* + 1$ and let us compute the difference between errors at two consecutive values of λ, say λT and $(\lambda+1)T$. To this aim, in Figure 1 we give a graphical representation of the behaviour of $\bar{u}(k)$, $\bar{x}(k)$ and $\underline{x}(k)$. From the figure, one can notice that in the two cases, when $\bar{u}$ is decreasing (figure on the left) and increasing (figure on the right), the maximum relative error is given respectively at λT and ad $\lambda T + \delta(\mathcal{G})$. Thus we compute the error in the case the input is increasing

$$\begin{aligned} e(\lambda T) &= \max_{i\in V} \frac{|x_i(\lambda T) - \bar{u}(\lambda T)|}{\bar{u}(\lambda T)} \\ &= \frac{\bar{u}((\lambda-1)T) - (1-\Pi)^T\bar{u}((\lambda-1)T)}{(1-\Pi)^T\bar{u}((\lambda-1)T)} \\ &= \frac{1}{(1-\Pi)^T} - 1 \,, \end{aligned} \tag{18}$$

and in the case the input is decreasing

$$\begin{aligned} e(\lambda T + \delta(\mathcal{G})) &= \max_{i \in V} \frac{|x_i(\lambda T + \alpha^{\delta(\mathcal{G})}) - \bar{u}((\lambda T + \alpha^{\delta(\mathcal{G})})|}{\bar{u}(\lambda T + \alpha^{\delta(\mathcal{G})})} \\ &= \frac{(1+\Pi)^{T\alpha^{\delta(\mathcal{G})}} \bar{u}(\lambda T) - \alpha^{\delta(\mathcal{G})} \bar{u}(\lambda T)}{(1+\Pi)^{T+\delta(\mathcal{G})} \bar{u}((\lambda - 1)T)} \\ &= 1 - \frac{\alpha^{\delta(\mathcal{G})}}{(1-\Pi)^{T+\delta(\mathcal{G})}} \; . \end{aligned} \tag{19}$$

Taking the maximum value between errors in (18) and (19) we obtain an upper-bound on the relative error for any time step greater than $\lambda^* T$, thus proving the theorem.

6 Numerical simulations

To illustrate the performance of the proposed protocol, simulation results are given in this section. The system monitoring the inputs is a distributed system which consists of 6 nodes, with interconnections given by graph in Figure 2, running the DMC protocol in Protocol 1. The choice of the graph is instrumental to show simulations in the worst case scenario, a line graph, where the information takes exactly $\delta(\mathcal{G}) = 5$ steps to flow through the graph.

Figure 3 shows evolution of the state variables (in red) when they are initialized at $x(0) = [0,\ 0.4,\ 0.8,\ 1.2,\ 1.6,\ 2]^T$, the inputs are constant at $u(k) = [0,\ 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1]^T$ with maximum $\bar{u} = 1$ (in blue) for all $k \geq 0$ and the tuning parameter α is 0.97. With respect to Theorem 1, $k^* = 0$ since the inputs are constant for $k \geq 0$, T can be chosen arbitrarily thus we chose $T = 0$, and accordingly to (6)

$$\bar{k} = \lceil log_\alpha\,(0.5) \rceil + \delta(\mathcal{G}) = 28 \, .$$

In fact in Figure 3 one can notice that after $\bar{k} = 28$ steps the system reaches a steady with relative error equal to (7), i.e., $e_i(k) \leq 1 - \alpha^{\delta(\mathcal{G})} = 1.413$, as shown in figure.

Figure 4 shows evolution of the state variables (in red) when they are initialized at $x(0) = [0,\ 0.4,\ 0.8,\ 1.2,\ 1.6,\ 2]^T$, the inputs are time-varying and initialized at $u(0) = [0,\ 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1]^T$ (the maximum in blue), the tuning parameter α is 0.97 and the relative change of the inputs is $\Pi = 9 \cdot 10^{-3}$. All inputs stay constant except for the 6-th component, which is time-varying with

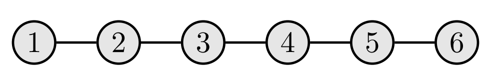

Fig. 2. Graph topology

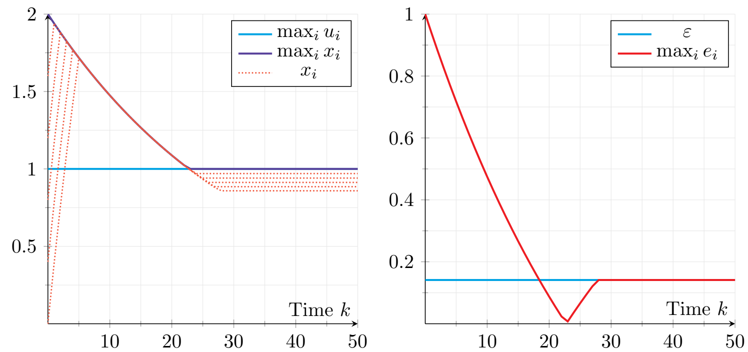

Fig. 3. Evolution of a MAS evolving according to (5), with constant input and random initialization.

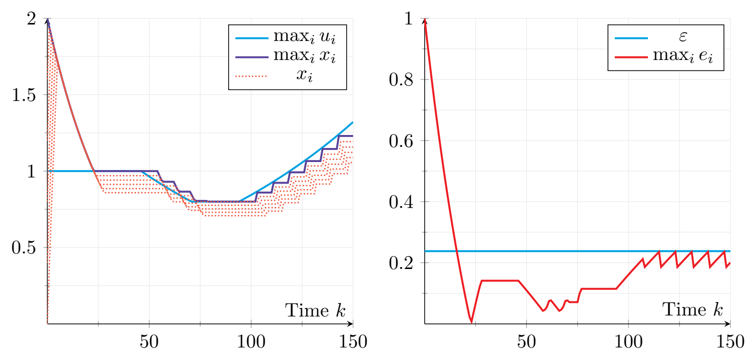

Fig. 4. Evolution of a MAS evolving according to (5), with time-varying input and initialization to the input.

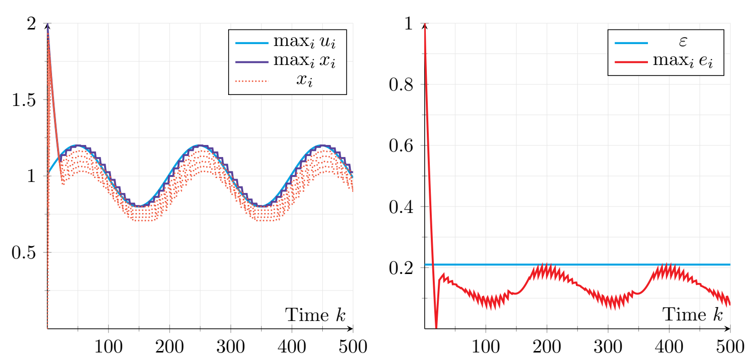

Fig. 5. Evolution of a MAS evolving according to (5), with sinusoidal input and initialization to the input.

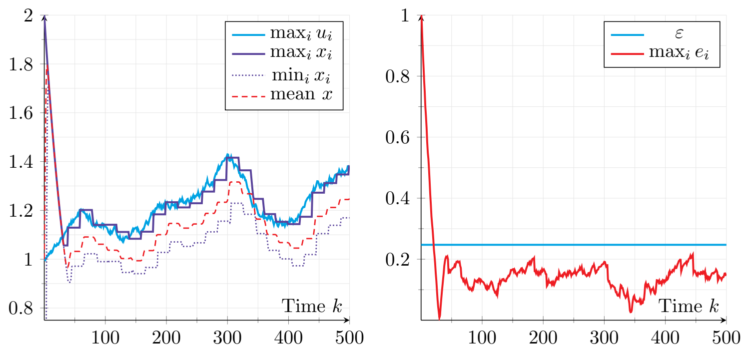

Fig. 6. Evolution of a large MAS evolving according to (5), with random input and initialization to the input.

respect to the following

$$u_6(k) \begin{cases} u_6(0) & \text{if } k < 48 \\ (1-\mathit{\Pi})u_6(0) & \text{if } k < 96 \\ (1+\mathit{\Pi})u_6(0) & \text{otherwise} \end{cases} . \tag{20}$$

As in the previous example, one can compute the convergence time which is $\bar{k} = 28$, as it is confirmed by the figure. Since $\mathit{\Pi} < 1 - \alpha$, by choosing T according to (9), i.e., $T = 7$, Theorem 2 holds true and thus, by substituting for the tuning parameters considered, there exists a time $\bar{k}$ such that it holds

$$\max_{i \in V} \frac{|x_i(k) - \bar{u}(k)|}{\bar{u}(k)} \leq 0.2357,$$

as shown in figure.

Figure 5 shows evolution of the state variables (in red) when they are initialized at $x(0) = [0,\ 0.4,\ 0.8,\ 1.2,\ 1.6,\ 2]^T$, the inputs are time-varying and initialized at $u(0) = [0,\ 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1]^T$ (the maximum in blue), the tuning parameter α is 0.97 and the relative change of the inputs is $\mathit{\Pi} = 64 \cdot 10^{-4}$. All inputs stay constant but the 6-component which is time-varying with respect to the following

$$u_6(k) = u_6(0) + 0.2 \sin\left(\frac{2k\pi}{200}\right) , \tag{21}$$

In this case, signals $u_i(k)$ are changing since the beginning, thus it makes no sense to think about computing a step k^* such that a steady state is reached. Nevertheless, since $\mathit{\Pi} < 1 - \alpha$, by choosing T according to (9), i.e., $T = 8$, Theorem 2 holds true and thus there exists a time $\bar{k}$ such that $\forall k \geq \bar{k}$ and $\forall i \in V$ it holds

$$e_i(k) \leq \max\left\{\frac{1}{(1-\mathit{\Pi})^T} - 1, 1 - \frac{\alpha^{\delta(\mathcal{G})}}{(1+\mathit{\Pi})^{T+\delta(\mathcal{G})}}\right\} = 0.2046,$$

as shown in figure.

Last simulation, shown in Figure 6, a different scenario is considered. The system monitoring the inputs is a large network of 100 nodes with random interconnections having diameter equal to $\delta(\mathcal{G}) = 9$. Figure 6 shows the evolution of the maximum among all inputs (in blue), the maximum and minimum among all states (in violet, respectively with solid and dotted lines) and finally the average state (in red). State variables are initialized at $x(0) = [0,\ 0.4,\ 0.8,\ 1.2,\ 1.6,\ 2]^T$, inputs are randomly time-varying such that

$$\frac{u(k+1)}{u(k)} \in [(1-\Pi), (1+\Pi)] \quad \forall k \geq 0$$

with relative change $\Pi = 10{\cdot}10^{-3}$ and initialized at $u(0) = [0, 0.2, 0.4, 0.6, 0.8, 1]^T$, while the tuning parameter α is 0.98. As in the previous example, signals $u_i(k)$ are changing since the beginning, thus the system wont reach a steady state and we check directly that conditions (9) holds true.

For instance, $\Pi < 1-\alpha$ is true and T must be chosen equals to 18. By applying Theorem 2 there exists a time $\bar{k}$ such that $\forall k \geq \bar{k}$ and $\forall i \in V$ it holds

$$e_i(k) \leq \max\left\{\frac{1}{(1-\Pi)^T} - 1, 1 - \frac{\alpha^{\delta(\mathcal{G})}}{(1+\Pi)^{T+\delta(\mathcal{G})}}\right\} = 0.2472,$$

as shown in figure.

7 Conclusions

We have proposed, and characterized in terms time and error convergence, a distributed protocol for multi-agent systems to effectively dealing with the problem of tracking the maximum of a set of positive time-varying input reference signals. Two strengths of the proposed protocol are the following: 1) the ability to track the maximum reference signal even it is strictly lower than all states variables; 2) the robustness to initialization, meaning that the protocol is ensured to works for any initialization of the state variables. A weakness of this protocol is that exact consensus is never reached, even with constant reference inputs, thus avoiding the chance to reach a zero error. In the view of this weakness, we aim to improve the proposed protocol by means of locally distributed and time-varying tuning parameters to ensure convergence to a consensus state.

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