Scale-free Estimation of Aggregated State in Linear Time-Invariant Systems

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58th IEEE Conference on Decision and Control Nice, France

Outline

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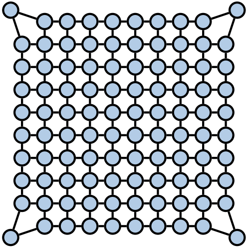
State Estimation



$$\Sigma : \begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} = C\mathbf{x} \end{cases}$$

$$\mathbf{x} \in \mathbb{R}^n, \quad \mathbf{y} \in \mathbb{R}^m$$

$$n \gg m$$

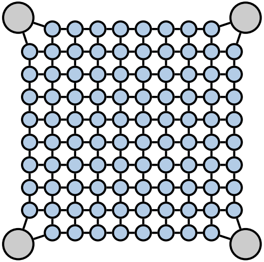
Problem

When and how can one asymptotically estimate the whole state $\mathbf{x}$?

Main issues:

- Sometimes estimating the whole state of a system is impossible.
- If possible, the estimation problem is computationally expensive.

Average State Estimation



$$\Sigma : \begin{cases} \begin{bmatrix} \dot{\mathbf{x}}_u \\ \dot{\mathbf{x}}_m \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_u \\ \mathbf{x}_m \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \mathbf{u} \\ \mathbf{y} = \begin{bmatrix} \mathbf{0}_{m \times k} & I_m \end{bmatrix} \begin{bmatrix} \mathbf{x}_u \\ \mathbf{x}_m \end{bmatrix} \end{cases}$$

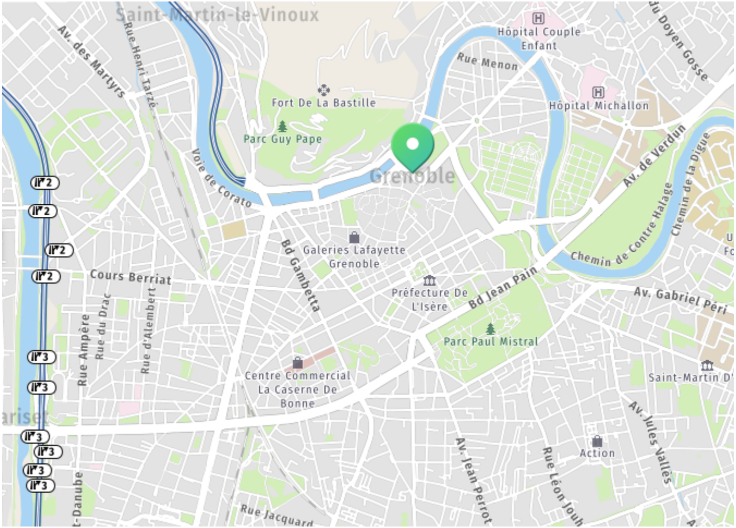
Problem

When and how can one asymptotically estimate the average state $x_{ave} = \frac{1}{k}\mathbf{1}^T\mathbf{x}_u$?

measured states $\mathbf{x}_m \in \mathbb{R}^m$

unmeasured states $\mathbf{x}_u \in \mathbb{R}^k$

Motivational Example: Monitoring Urban Traffic



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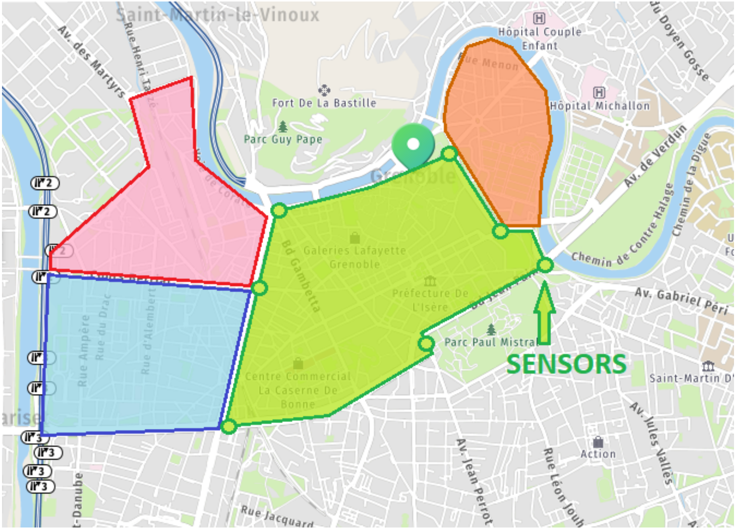
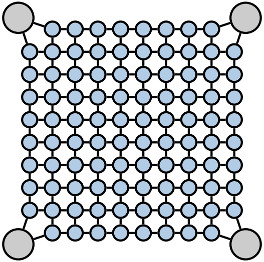
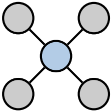


Our approach



$$\mathbf{z} = P\mathbf{x}$$

Projection



$$\underbrace{\begin{bmatrix} x_{ave}(t) \\ \mathbf{x}_m(t) \end{bmatrix}}_{\mathbf{z}} = \underbrace{\begin{bmatrix} \frac{1}{k}\mathbf{1}_k^T & 0 \\ 0 & I_m \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} \mathbf{x}_u(t) \\ \mathbf{x}_m(t) \end{bmatrix}}_{x}$$

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} = C\mathbf{x} \end{cases}$$

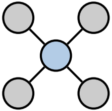
Projection

$$\boldsymbol{\sigma} = \mathbf{x}_u - \mathbf{1}x_{ave}$$

$$\Omega : \begin{cases} \dot{\mathbf{z}} = A'\mathbf{z} + B'\mathbf{u} + F\boldsymbol{\sigma} \\ \mathbf{y} = C'\mathbf{z} \\ 0 = \mathbf{1}^T\boldsymbol{\sigma} \end{cases}$$

Estimate of $\mathbf{z}$ is equivalent to average estimation of $\mathbf{x}$

Main contributions



$$\Omega : \begin{cases} \dot{\mathbf{z}} = A'\mathbf{z} + B'\mathbf{u} + F\boldsymbol{\sigma} \\ \mathbf{y} = C'\mathbf{z} \\ 0 = \mathbf{1}^T\boldsymbol{\sigma} \end{cases}$$

$$A' = \begin{bmatrix} \frac{1}{k}\mathbf{1}^T A_{11}\mathbf{1} & \frac{1}{k}\mathbf{1}^T A_{12} \\ A_{21}\mathbf{1} & A_{22} \end{bmatrix}, \quad B' = \begin{bmatrix} \frac{1}{k}\mathbf{1}^T B_1\mathbf{1} \\ A_2 \end{bmatrix}, \quad F = \begin{bmatrix} \frac{1}{k}\mathbf{1}^T A_{11}\mathbf{1} \\ A_{21} \end{bmatrix}, \quad C' = \begin{bmatrix} \mathbf{0}_m & I_m \end{bmatrix}$$

Our Problem

When and how can one asymptotically estimate the state $\mathbf{z}$ regardless of the unknown input $\boldsymbol{\sigma}$?

- When: We provide a necessary and sufficient condition for the existence of an observer $\widehat{\Omega}$ for the system Ω
- How: We provide two different observer designs

Outline

Necessary and sufficient condition

Theorem 1

Consider a system Σ, its lower order projection Ω and an observer $\hat{\Omega}$ in the following form

$$\widehat{\Omega} = \begin{cases} \dot{\mathbf{w}} = M\mathbf{w} + K\mathbf{y} + NG\mathbf{u} \\ \hat{\mathbf{z}} = \mathbf{w} + L\mathbf{y} \end{cases}, \quad \mathbf{w} \in \mathbb{R}^{m+1} \quad .$$

It is possible to design M, K, N, L such that the estimation error $\mathbf{e}(t) := \mathbf{z}(t) - \hat{\mathbf{z}}(t)$ converges to 0 as $t \to \infty$ at an arbitrary rate if and only if

$$rank \begin{bmatrix} \mathbf{1}^T \\ \mathbf{1}^T A_{11} \\ A_{21} \end{bmatrix} = rank \begin{bmatrix} A_{21} \end{bmatrix} . \quad (1)$$

Proof Sketch

Having chosen

$$N = I - LC', \qquad M = NA' - K_1 C' \,, \tag{2}$$
$$K = K_1 + K_2, \qquad K_2 = ML \,,$$

the error of the observer and its dynamics can be written as

$$\mathbf{e}(t) = \hat{\mathbf{z}}(t) - \mathbf{z}(t) \quad , \tag{3}$$

$$\dot{\mathbf{e}}(t) = \dot{\hat{\mathbf{z}}}(t) - \dot{\mathbf{z}}(t) = M\mathbf{e}(t) + NF\boldsymbol{\sigma}(t) \quad . \tag{4}$$

To ensure global asymptotical stability of $\mathbf{e}(t)$, one has to show:

1. $NF\boldsymbol{\sigma}(t) = 0$ for all t.
2. $\lambda \in eig(M)$ are such that $\Re\{\lambda\} < 0$

To ensure $\|\mathbf{e}(t)\| \to 0$ as $t \to \infty$ at an 'arbitrary' rate, one has to show, in addition to 1 and 2 above, that $eig(M)$ can be assigned arbitrarily.

Design Procedures

Design 1

Design the observer $\widehat{\boldsymbol{\Omega}}$ such that

$$\begin{aligned} L &= \begin{bmatrix} \frac{1}{k}\mathbf{1}^T A_{11} - v_1 \mathbf{1}^T \\ A_{21} - \mathbf{v}_2 \mathbf{1}^T \end{bmatrix} A_{21}^{+}, \\ K_1 &= \text{place}(NA', C', [\lambda_1, \ldots, \lambda_{m+1}]), \end{aligned} \tag{5}$$

where $\lambda_i \in \mathbb{C}_{<0}$, for $i = 1, \cdots, m+1$, are the desired eigenvalues of M, "place" is the classic pole-placement algorithm[1], $v_1 \in \mathbb{R}$ and $\mathbf{v}_2 \in \mathbb{R}^m \setminus \{0_m\}$ are arbitrary and

$$N = I - LC', \quad M = NA' - K_1 C',$$

$$K = K_1 + K_2, \quad K_2 = ML.$$

[1] **Kautsky et al.**, *Robust pole assignment in linear state feedback* (1985)

Design Procedures

Design 2

Design the observer $\widehat{\boldsymbol{\Omega}}$ such that

$$\begin{aligned} L &= \begin{bmatrix} (\frac{1}{k}\mathbf{1}^T A_{11} - v_1 \mathbf{1}^T) A_{21}^{+} \\ I_m \end{bmatrix}, \\ K_1 &= \begin{bmatrix} \frac{1}{k}\mathbf{1}^T A_{12} - \boldsymbol{\ell}_1^T A_{22} \\ -diag[\lambda_2, \cdots, \lambda_{m+1}] \end{bmatrix}, \\ v_1 &= \frac{\lambda_1 - \frac{1}{k}\mathbf{1}^{\mathbf{T}} \mathbf{A}_{\mathbf{11}} (\mathbf{I} - \mathbf{A}_{\mathbf{21}}^{+} \mathbf{A}_{\mathbf{21}}) \mathbf{1}}{\mathbf{1}^{\mathbf{T}} \mathbf{A}_{\mathbf{21}}^{+} \mathbf{A}_{\mathbf{21}} \mathbf{1}}, \end{aligned} \tag{6}$$

where $\lambda_i \in \mathbb{R}_{<0}$, for $i = 1, \cdots, m+1$ are the desired eigenvalues of M and

$$N = I - LC', \quad M = NA' - K_1 C',$$

$$K = K_1 + K_2, \quad K_2 = ML.$$

With this design M is diagonal, therefore it yields a reduced-order observer of dimension equal to 1.

Outline

Bounded Error

Theorem 2

Consider a system Σ, its lower order projection Ω and an observer $\widehat{\Omega}$ in the following form

$$\widehat{\Omega} = \begin{cases} \dot{\mathbf{w}}(t) = M\mathbf{w}(t) + K\mathbf{y}(t) + NG\mathbf{u}(t) \\ \hat{\mathbf{z}}(t) = \mathbf{w}(t) + L\mathbf{y}(t) \end{cases}, \quad \mathbf{w} \in \mathbb{R}^{m+1} \quad .$$

It is possible to design M, K, N, L such that the estimation error $\mathbf{e}(t) := \mathbf{z}(t) - \hat{\mathbf{z}}(t)$ is bounded as $t \to \infty$ if one of the following holds

1. eig$(A) \subset \mathbb{C}_{\leq 0}$ and $\int_0^\infty \|\mathbf{u}(t)\| dt < \infty$.
2. eig$(A) \subset \mathbb{C}_{<0}$ and $\|\mathbf{u}(t)\| < \infty$ for all $t \in \mathbb{R}_{\geq 0}$.

Proof Sketch

$$\hat{\mathbf{z}}(t) - \mathbf{z}(t) = \mathbf{e}(t) = e^{Mt}\mathbf{e}(0) + \int_0^t e^{M(t-\tau)} NF\boldsymbol{\sigma}(\tau)\, d\tau.$$

$$\begin{aligned}
\lim_{t\to\infty} \|\mathbf{e}(t)\| &\leq \lim_{t\to\infty} \left\| \int_0^t e^{M(t-\tau)} NF\boldsymbol{\sigma}(\tau) \right\| d\tau \\
&\leq \lim_{t\to\infty} \int_0^t \left\| e^{M(t-\tau)} NF\boldsymbol{\sigma}(\tau) \right\| d\tau \\
&\leq \lim_{t\to\infty} \int_0^t \left|\left|\left| e^{M(t-\tau)} \right|\right|\right| \|NF\boldsymbol{\sigma}(\tau)\|\, d\tau. \\
&\leq \left[\max_{t\geq 0} \|NF\boldsymbol{\sigma}(\mathbf{t})\| \right] \left[\lim_{t\to\infty} \int_0^t \left|\left|\left| e^{M(t-\tau)} \right|\right|\right| d\tau \right] \\
&\leq |||V^{-1}|||\,|||V|||\, \frac{|||N(\lambda^*)FJ|||}{\lambda^*} \max_{t\geq 0} \|\mathbf{x_u}(\mathbf{t})\|,
\end{aligned}$$

$\lambda^* = \min_{\lambda\in \mathsf{eig}(M)} |\mathsf{Re}\{\lambda\}| > 0$

Outline

Compartmental System

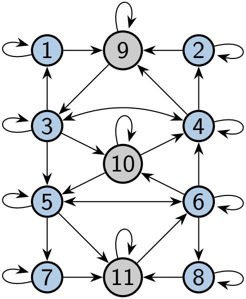


Figure: Example 1

$$\begin{cases} a_{ii} = & -\sum_{h=1,h\neq i} a_{hi} \\ a_{ij} = & 1 \text{ if there is edge } (j,i) \\ a_{ij} = & 0 \text{ otherwise} \end{cases}$$

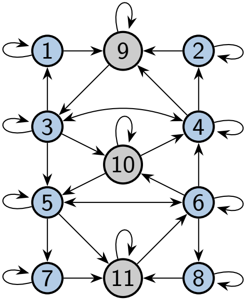
$$B = C^T, \quad \mathbf{u}(t) = 10[\sin t \quad \sin 10t \quad \sin 20t]^T.$$

$$A_{21} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\mathbf{1}^T A_{11} = -\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\text{eig}(M) = \{-0.75, -1, -2, -3\}$$

Exact Estimation



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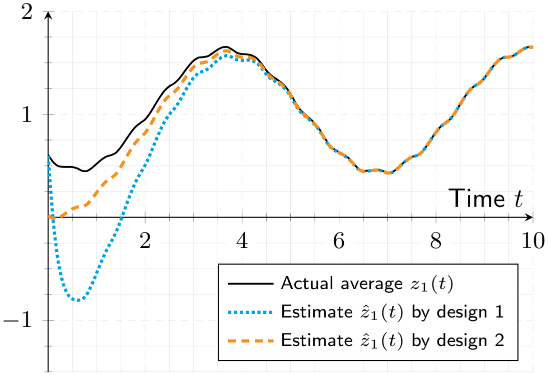

Figure: Average state estimation for Example 1.

Reaction-Diffusion System

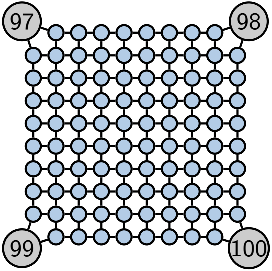

Figure: Example 2

$$\begin{cases} a_{ii} = & -r_i = -0.2 \\ a_{ij} = & 1 \text{ if there is edge } (j,i) \\ a_{ij} = & 0 \text{ otherwise} \end{cases}$$

- $u_1(t) = \sin(0.05t)$ applied at nodes 97 and 98;
- $u_2(t) = \sin(t)$ applied at nodes 99 and 100;
- $u_3(t) = 0.01$ applied at the remaining boundary nodes of the grid

Design 1: $\text{eig}(M) = \{-0.5, -1, -2, -3, -4\}$

Design 2: $\text{eig}(M) = \{\mathbf{-0.0237}, -1, -2, -3, -4\}$

Approximate Estimation

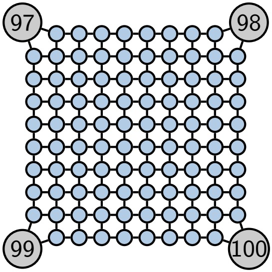

Figure: Example 2

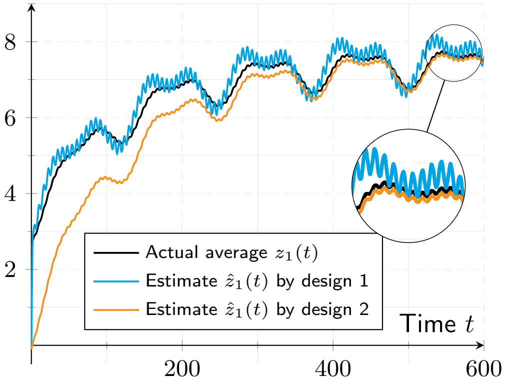

Figure: Average state estimation for Example 2.

Outline

Conclusions and future perspectives

Conclusions

- A necessary and sufficient condition for the existence of an average state observer is provided
- Two designs for the observer are provided
  - Design 2 yields an observer of dimension 1 with minimum error
- Complexity and error do not scale with the system.

Ongoing work

- Extension to multiple clusters

  Niazi, Canudas-de-Wit, Kibangou, *Average state estimation in large-scale clustered network systems*", TCNS 2019.
- Clustering algorithms for average estimation

  Niazi, Cheng, Canudas-de-Wit, Scherpen, "*Structure-based clustering for model reduction of large-scale networks*", CDC 2019.

Future work

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Thank you for your attention

Questions?

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This work was supported in part by the European Research Council through the European Union's Horizon 2020 Research and Innovation Programme (Scale-FreeBack) under Grant ERC-AdG no. 694209.