Distributed Fiedler Vector Estimation with Application to Desynchronization of Harmonic Oscillator Networks

Diego Deplano[†*], M. Franceschelli[*], A. Giua[*], L. Scardovi[**]

[*]Department of Electrical and Electronic Engineering, University of Cagliari, Italy.
[**]Department of Electrical and Computer Engineering, University of Toronto, Canada.

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Outline

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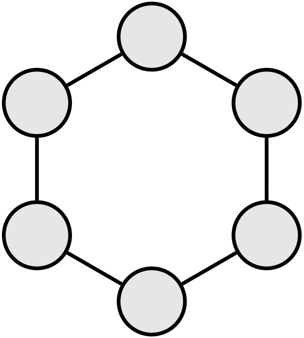
From synchronization...

Examples of biological oscillators:

- **Fireflies**
- Neuron firing
- Circadian rhythms
- Signal transduction,
- Cell cycles
- ...

The idea of synchronization:

- Oscillators execute tasks at the same time



From synchronization...

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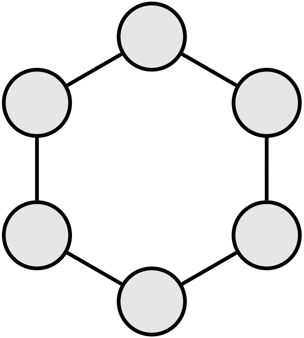

The idea of synchronization:

- Oscillators execute tasks at the same time

...to desynchronization

Examples of biological oscillators:

- **Fireflies**
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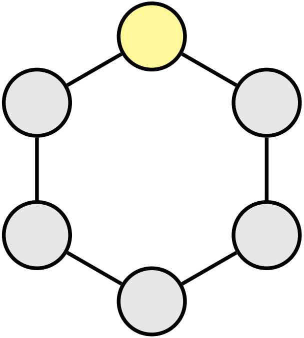
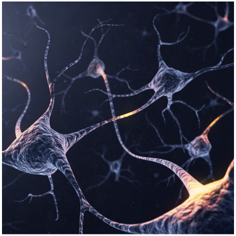
The idea of **de**synchronization:

- Oscillators execute tasks
  as far away as possible from all others



Why desynchronization?



Neuronal networks

Wireless sensor networks

Mechanical networks

Electrical networks

Main contribution

Problem of interest

Given a network of harmonic oscillators with diffusive coupling, design local protocol to achieve desynchronization in a distributed manner.

Outline of the contributions

1. Design of a distributed control action to achieve desynchronization in the network;
2. Employment of the proposed protocol to estimate the Fiedler vector in networks of single integrator agents.

Literature on harmonic oscillators

Literature on desynchronization:

- ?

Literature on synchronization:

- W. Ren, "Synchronization of coupled harmonic oscillators with local interaction," Automatica, vol. 44, no. 12, pp. 3195–3200, 2008.
- L. Scardovi and R. Sepulchre, "Synchronization in networks of identical linear systems," Automatica, vol. 45, pp. 2557–2562, Nov. 2009.
- Su, Housheng, Xiaofan Wang, and Zongli Lin. "Synchronization of coupled harmonic oscillators in a dynamic proximity network." Automatica 45.10 (2009): 2286-2291.
- X. Liu and T. Iwasaki, "Design of coupled harmonic oscillators for synchronization and coordination," IEEE Transaction on Automatic Control, vol. 62, no. 8, pp. 3877–3889, 2017.
- S. E. Tuna, "Synchronization of harmonic oscillators under restorative coupling with applications in electrical networks," Automatica, vol. 75, pp. 236–243, 2017.
- Q. Song, F. Liu, J. Cao, A. Vasilakos, and Y. Tang, "Leader-following synchronization of coupled homogeneous and heterogeneous harmonic oscillators based on relative position measurements," IEEE Transaction on Control of Network Systems, vol. 6, no. 1, pp. 13–23, 2018.
- H. Zhang, Q. Wu, and J. Ji, "Synchronization of discretely coupled harmonic oscillators using sampled position states only," IEEE Transaction on Automatic Control, vol. 63, no. 11, pp. 3994–3999, 2018.
- S. Baldi and P. Frasca, "Leaderless synchronization of heterogeneous oscillators by adaptively learning the group model," IEEE Transaction on Automatic Control, vol 65, no. 11, pp. 412-418, 2019.
- and many others

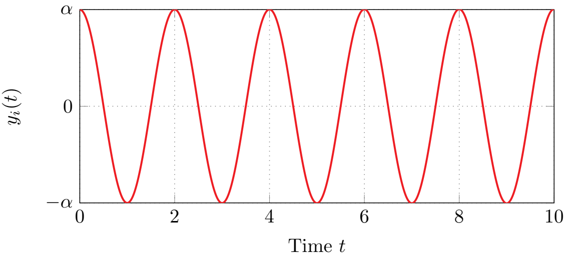
Outline

Harmonic oscillators

Harmonic oscillators are second-order dynamical systems exhibiting periodic motion.

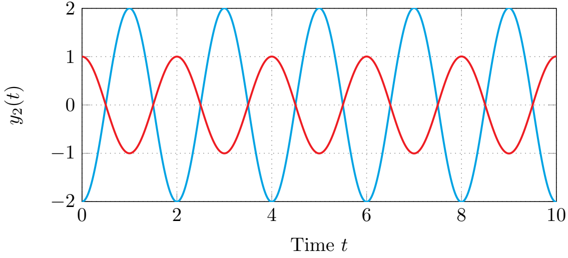
$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i \\ y_i &= Cx_i \end{aligned} \qquad \text{with } A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}, \quad B = C^\intercal = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

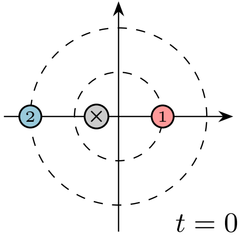
For instance, consider a single oscillator with $\omega = \pi$ and $x(0) = \begin{bmatrix} 0 & \alpha \end{bmatrix}^\intercal$,



General order parameter

$$y_i(t) = M_i \cos(\omega t + \theta_i) \Rightarrow y_i = \Re\left\{M_i e^{j(\omega t + \theta_i)}\right\}$$





Assuming a rotating frame with angular frequency ω, define the centroid

$$Re^{j\phi} = \frac{1}{\sum_{i=1}^{n} M_i} \sum_{i=1}^{n} M_i e^{j\theta_i}, \qquad \sum_{i=1} n > 0, \tag{1}$$

where R measures the level of desynchronization of the network.

Network of harmonic oscillators

The i-th harmonic oscillator with $i \in \{0, \ldots, n\} = V$ in the network has dynamics

$$\dot{x}_i = Ax_i + B(u_i^d + u_i^c) \qquad u_i^d = \sum_{j \in \mathcal{N}_i} (y_j - y_i), \qquad u_i^c = ?$$
$$y_i = Cx_i$$

where

- $\mathcal{G} = (V, E)$ describe the both coupling and communication networks, and it is assumed to be undirected;
- u_i^d denotes the diffusive coupling among oscillators;
- u_i^c denotes the local control action to be designed;

Desynchronization measure

Desynchronization measure

Consider a network of n identical oscillators. Denoting with $y_i^{ss}(t)$ the steady state of the oscillators, the network is said to have achieve desynchronization if

$$R = 0 \Leftrightarrow \sum_{i=1}^{n} y_i^{ss}(t) = 0, \tag{2}$$

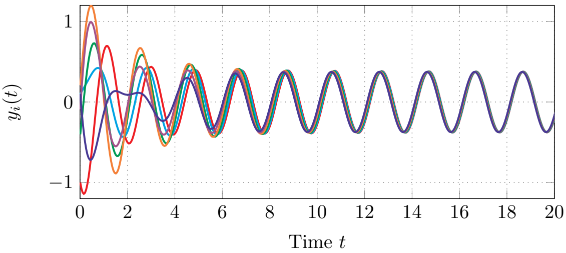
i.e., the collective steady-state output dynamics is non-null with zero mean or, equivalently, the centroid is at the origin of the Complex Plane.

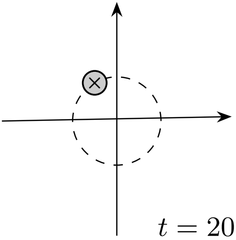
Diffusive coupling leads to synchronization

Harmonic oscillators are assumed to be diffusively coupled according to a connected undirected graph $\mathcal{G}$.

$$\dot{x}_i = Ax_i + B(u_i^d + u_i^c) \qquad u_i^d = \sum_{j \in \mathcal{N}_i} (y_j - y_i)$$
$$y_i = Cx_i$$

Consider an oscillators network with no local control action $u_i^c = 0$,



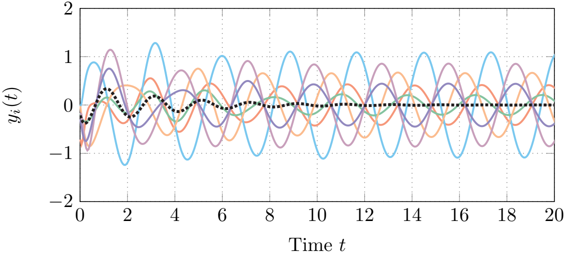


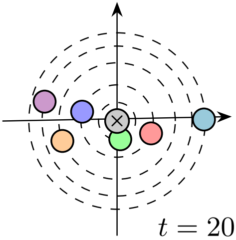
What leads to desynchronization?

Harmonic oscillators are assumed to be diffusively coupled according to a connected undirected graph $\mathcal{G}$.

$$\dot{x}_i = Ax_i + B(u_i^d + u_i^c) \qquad u_i^d = \sum_{j\in\mathcal{N}_i}(y_j - y_i)$$
$$y_i = Cx_i$$

How to design a control action $u_i^c = ?$ to achieve desynchronization?





Outline

Main result

Desynchronization of harmonic oscillators

Consider a network of n diffusively coupled harmonic oscillators driven by

$$
\begin{aligned}
u_i^c &= -\alpha v_i + \lambda_2 y_i \\
\dot{v}_i &= \dot{y}_i + \alpha(y_i - v_i) + \gamma \sum_{j \in \mathcal{N}_i} (z_j - z_i) \\
\dot{z}_i &= \gamma v_i
\end{aligned}
$$

If graph $\mathcal{G}$ is connected and

$$
\alpha > \max\{\omega^2, 2\lambda_2\}, \quad \gamma = \sqrt{(2\alpha^2 + \omega^2)/(2\lambda_2)},
$$

then the network achieve desynchronization almost globally.

The idea behind the proof

$$
\begin{aligned}
u_i &= \overbrace{\sum_{j\in\mathcal{N}_i}(y_j - y_i)}^{u_i^d}\ \overbrace{-\alpha v_i + \lambda_2 y_i}^{u_i^c} \\
\dot{v}_i &= \dot{y}_i + \beta(y_i - v_i) + \gamma \sum_{j\in\mathcal{N}_i}(z_j - z_i) \\
\dot{z}_i &= \gamma v_i
\end{aligned}
$$

The quantity v_i is a distributed estimator of the average output $\mathbf{1}^\intercal y = \sum_{i=1}^n y_i$ and thus at steady state the input reads

$$
u = \underbrace{-Ly - \alpha \mathbf{1}\mathbf{1}^\intercal y + \lambda_2 y}_{-My}
$$

The idea behind the proof

$$u = -\underbrace{(L + \alpha \mathbf{1}\mathbf{1}^\intercal - \lambda_2 I)}_{M}\, y$$

The eigenvalues of matrix L satisfy $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.
The construction of matrix M consists of two steps:

1. *Matrix inflation* by $\alpha \mathbf{1}\mathbf{1}^\intercal$, with $\alpha > \lambda_2/n$ which makes the eigenvalue λ_1 shifting forward without changing the eigenvectors.
2. *Eigenvalues shifting* by $-\lambda_2 I$ which makes all eigenvalues shifting backward without changing the eigenvectors.

Therefore,

- The smallest eigenvalue of M is null.
- The associated eigenvector is the Fiedler vector v_2.

Main result

Ditributed Fiedler vector estimation

Consider a network of n single integrators $\dot{y}_i = u_i$ driven by

$$u_i = \sum_{j \in \mathcal{N}_i} (y_j - y_i) - \alpha v_i + \lambda_2 y_i$$

$$\dot{v}_i = \dot{y}_i + \beta(y_i - v_i) + \gamma \sum_{j \in \mathcal{N}_i} (z_j - z_i)$$

$$\dot{z}_i = \gamma v_i$$

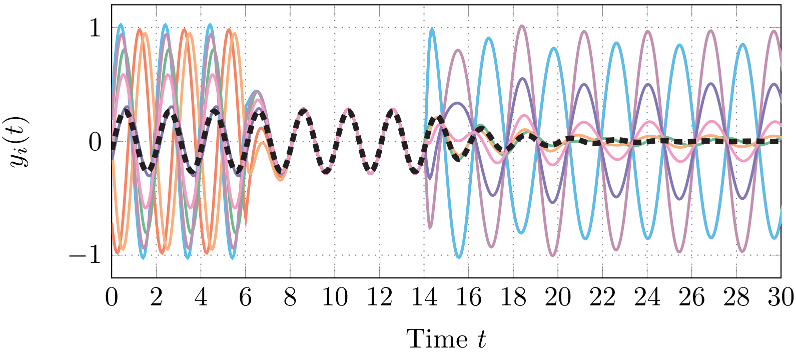
If graph $\mathcal{G}$ is connected, $\alpha > \lambda_2$, $\beta > 0$, $\gamma > 0$, then $y(t)$ converges almost globally to a state proportional to the Fiedler vector of graph $\mathcal{G}$, i.e.,

$$\lim_{t \to \infty} y(t) = c v_2, \qquad c \in \mathbb{R}.$$

Outline

Desynchronization of harmonic oscillators

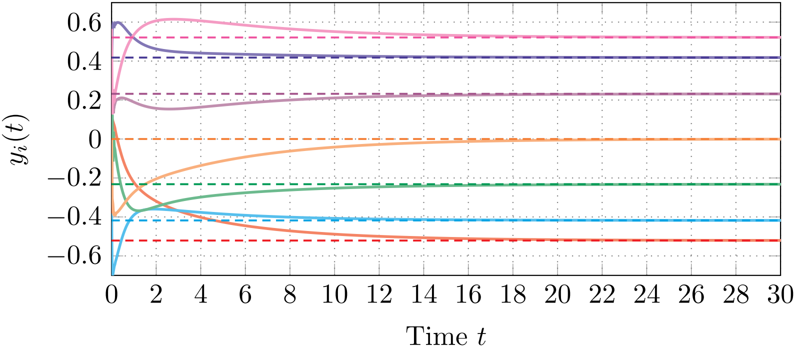
$$\omega = \pi, \quad \alpha = 7.7, \quad \gamma = 3.03.$$



- At $t = 6$ the diffusive coupling is activated;
- At $t = 14$ the proposed control action is activated.

Fiedler vector estimation

$$\omega = \pi, \quad \alpha = 7.7, \quad \gamma = 3.03.$$



Outline

Conclusions and future directions

Conclusions:

- A formal definition of desynchronization for harmonic oscillators has been proposed;
- A distributed control action to achieve desynchronization in networks of harmonic oscillators has been designed;
- The proposed protocol is shown to be able to estimate the Fiedler vector in networks of single integrators.

Future directions:

- Design a dynamic estimator for λ_2
- Time-varying topologies
- Heterogeneous oscillators

Thank you for your attention

email:
diego.deplano@unica.it

D. Deplano, M. Franceschelli, A. Giua, L. Scardovi, "*Distributed Fiedler Vector Estimation with Application to Desynchronization of Harmonic Oscillator Networks*", IEEE Control Systems Letters (L-CSS), 2020.