A unified approach to solve the dynamic consensus on the average, maximum, and median values with linear convergence

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***Abstract*— This manuscript proposes novel distributed algorithms for solving the dynamic consensus problem in discrete-time multi-agent systems on three different objective functions: the average, the maximum, and the median. In this problem, each agent has access to an external time-varying scalar signal and aims to estimate and track a function of all the signals by exploiting only local communications with other agents. By recasting the problem as an online distributed optimization problem, the proposed algorithms are derived based on the distributed implementation of the alternating direction method of multipliers (ADMM) and are thus amenable to a unified analysis technique. A major contribution is that of proving linear convergence of these ADMM-based algorithms for the specific dynamic consensus problems of interest, for which current results could only guarantee sub-linear convergence. In particular, the tracking error is shown to converge within a bound, whereas the steady-state error is zero. Numerical simulations corroborate the theoretical findings, empirically show the robustness of the proposed algorithms to re-initialization errors, and compare their performance with that of state-of-the-art algorithms.**

I. INTRODUCTION

In the context of multi-agent systems, significant efforts have been made to develop distributed algorithms that can solve the *dynamic consensus problem*. This problem involves designing local interaction rules that can drive agents to agree on a common value, which is determined by local time-varying signals they observe, relying solely on local computations and peer-to-peer communications. While the *average* function has garnered a lot of attention [1]–[4], with a comprehensive literature review provided by Kia *et al* [5], other relevant functions such as the *maximum* [6]–[8] and *median* [9], [10] functions have received less attention. Dynamic consensus has practical applications in network parameter estimation and control [11]–[13], resilience against faults and malicious attacks [14], [15], formation control, and containment in multi-robot systems [16]–[18], among others.

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Main contribution. This manuscript presents a unified approach to solving the dynamic consensus problem for average, maximum, and median functions in the discrete-time framework, which differs from ad-hoc approaches considered in the current literature. Numerical simulations reveal that the proposed algorithms extend the state-of-the-art in different ways:

- Better trade-off between convergence rate and tracking error for the dynamic average consensus, when compared to [5] and [19];
- Higher robustness to unexpected spikes in the variation of the inputs for the dynamic max consensus, when compared to [6];
- It is the first protocol that solves the dynamic median consensus in the case of heterogeneous inputs since the protocol in [10] has been shown to fail in this scenario.

Our approach takes the perspective of *online optimization*, where the goal is that of solving a streaming sequence of problems in real-time [20]–[22]. These dynamic consensus problems can be formulated as sequences of convex distributed optimization problems of a similar kind, from which we derive online protocols that are based on the distributed operator theoretical (DOT) implementation of the *relaxed alternating direction method of multipliers* (R-ADMM), which is called the DOT-ADMM [22]; its theoretical formulation has been recently described in [23] and it has been applied to online learning in [22].

Although ADMM-based algorithms are known to converge sub-linearly for convex problems [24], we prove that the DOT-ADMM achieves linear convergence for these specific dynamic consensus problems, which is a major contribution of the paper. We achieve this by deriving the explicit implementation of its updates and by showing that they are globally metric subregular [25, Section IV.B]. This, together with the averagedness property guaranteed by the convexity of the problem, turns out to be sufficient for global linear convergence without the strong convexity assumption.

Related literature. We briefly review state-of-the-art protocols that work in the discrete-time framework and share characteristics with our proposed protocols, such as robustness to re-initialization errors and the ability to handle arbitrary and unbounded reference signals with bounded derivatives.

(Average) - The protocol proposed by Montijano *et al.* in [26], tracks asymptotically the time-varying average with bounded steady-state error by means of damping factors but needs knowledge of the reference signals' derivative. In contrast, the multi-stage dynamic average consensus (MSDAC)

protocol proposed by Franceschelli *et al.* [19] does not require such knowledge by employing a multi-stage scheme of protocol by Freeman *et al.* [3], and converges asymptotically with a steady-state error that decreases geometrically with the number of stages.

(Maximum) - Among the three different protocols we have proposed to solve the dynamic version of the max-consensus problem [6], [7], [27], the best performing is the self-tuning dynamic max-consensus (STDMC) protocol, whose steady-state error can be made arbitrarily small and is decoupled by design from the tracking error which can be traded-off for improved convergence time. All these protocols converge in finite time but require the knowledge of a bound on the reference signals' derivative.

(Median) - The only protocol in the current literature that enables the distributed tracking of time-varying median value of locally measured signals in the discrete-time framework has been developed by Vasiljevic *et. al* in [10], which we refer to as the dynamic median consensus (DMEC) protocol, building upon [28]. The protocol of Yu *et al.* [29], while relevant, only considers essentially constant reference signals affected by decaying bias and white noise.

Structure of the paper. Notation and preliminaries are presented in Section II. The dynamic consensus problems on the average, maximum, and minimum function are formulated as distributed time-varying optimization problems in Section III, while operator-theoretical protocols to solve them are derived and characterized in Section IV. Section V provides numerical simulations substantiating the theoretical results and a comparison with state-of-the-art protocols.

II. Notation and Preliminaries

Matrices $M \in \mathbb{R}^{n\times n}$ are denoted by uppercase letters, vectors $\boldsymbol{v} \in \mathbb{R}^n$ by boldface bold letters, scalars $s \in \mathbb{R}$ by lowercase nonbold letters, while sets and spaces $\mathcal{S}$ are denoted by uppercase calligraphic letters. The vectors of ones and zeros are denoted by $\mathbf{1}$ and $\mathbf{0}$, respectively.

A. Networks and graphs

We consider networks modeled by graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, n\}$ with $n \in \mathbb{N}$ is the set of *nodes*, and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is the set of *edges* connecting the nodes. A graph is said to be *undirected* if for any edge $(i, j) \in \mathcal{E}$ there is also $(j, i) \in \mathcal{E}$. A path between two nodes $i, j \in \mathcal{V}$ is a sequence of consecutive edges $\pi_{ij} = (i, p), (p, q), \dots, (r, s), (s, j)$ where each successive edge shares a node with its predecessor. An undirected graph $\mathcal{G}$ is said to be *connected* if there exists a path π_{ij} between any pair of nodes $i, j \in \mathcal{V}$. The set of neighbors of the i-th node is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$.

B. Rate of convergence

A sequence $s(k)$ is said to converge to an interval $[a, b] \subset \mathbb{R}$ if there is $k^* \in \mathbb{R} \cup \{\infty\}$ such that

$$\lim_{k\to k^*} s(k) \in [a, b], \quad \text{and} \quad s(k) \in [a, b] \text{ for } \forall k \geq k^*.$$

The convergence occurs in finite time if $k^* \in \mathbb{R}$ and asymptotically if $k^* = \infty$. Moreover, if $a = b = s^*$, $s(k)$ is said to converge to the value s^*. We denote the distance from the sequence $s(k)$ to the interval $[a, b]$ by

$$|s(k) - [a, b]| = \max\{0, a - s(k), s(k) - b\}.$$

Definition 1: *Let $s(k)$ be a sequence converging to $[a, b]$ at k^*. The convergence is Q-linear if there is $\mu \in (0, 1)$ such that*

$$\lim_{k\to k^*} \frac{|s(k) - [a, b]|}{|s(k-1) - [a, b]|} = \mu. \tag{1}$$

The convergence is R-linear if there is a sequence $q(k)$ that converges Q-linearly to zero such that

$$|s(k) - [a, b]| \leq q(k), \qquad \forall k \in \mathbb{N}. \tag{2}$$

III. Problem statement

Consider a network of n agents modeled as discrete-time $k \in \mathbb{N}$ dynamical systems with scalar state $x_i(k) \in \mathbb{R}$ for $i = 1, \dots, n$. The i-th agent has access to a scalar time-varying reference signal $u_i(k) \in \mathbb{R}$ and interacts with other agents according to an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The agents can only exchange information about their states $x_i(k)$ but not about the reference signals $u_i(k)$ they have access to, which are assumed to have bounded derivatives but may be unbounded in value, as formalized next.

Assumption 1: *There exists a finite bound $\sigma \in [0, \infty)$ such that all reference signals observed by the agents satisfy*

$$|u_i(k + 1) - u_i(k)| \leq \sigma, \quad \forall i = 1, \dots, n.$$

The *dynamic consensus problem* consists in the design of proper local interaction rules to enable the tracking of a function $\mathrm{obj}(\boldsymbol{u}(k))$ of the time-varying reference signals $u_i(k) \in \mathbb{R}$ stacked into $\boldsymbol{u}(k) \in \mathbb{R}^n$. In particular, we are mainly interested in the following three functions: the *average* and *maximum* functions, which are defined by

$$\mathrm{avg}(\boldsymbol{u}(k)) = \frac{1}{n}\sum_{i=1}^{n} u_i(k), \quad \max(\boldsymbol{u}(k)) = \max_{i=1,\dots,n} u_i(k),$$

and the *median* function, which, letting $u_1(k), \dots, u_n(k)$ be sorted in ascending order, is defined by

$$\mathrm{med}(\boldsymbol{u}(k)) = \begin{cases} u_{\frac{n+1}{2}}(k) & \text{if } n \text{ is odd} \\ \frac{1}{2}(u_{\frac{n}{2}}(k) + u_{\frac{n}{2}+1}(k)) & \text{if } n \text{ is even} \end{cases}.$$

Letting $\boldsymbol{x}(k)$ be the vector stacking all agents' states and $\mathrm{obj} \in \{\mathrm{avg}, \max, \mathrm{med}\}$ be the function to be tracked, we will characterize the performance of the proposed protocols in terms of the tracking error $e(k) = \|\boldsymbol{x}(k) - \mathrm{obj}(\boldsymbol{u}(k))\mathbf{1}\|$.

A. Optimization problem description

The dynamic consensus problems on average, maximum, and median functions can be recast as distributed time-varying optimization problems of the following type

$$\begin{aligned} \boldsymbol{x}^*(k) = \operatorname*{argmin}_{x_1,\dots,x_n} \ & \sum_{i=1}^{n} \frac{1}{p}|x_i - u_i(k)|^p \\ \text{s.t.} \ & x_i = x_j \qquad \forall\, (i, j) \in \mathcal{E} \\ & x_i \in \mathcal{X}_{i,k} \quad \ \forall\, i \in \mathcal{V}. \end{aligned} \tag{3}$$

Indeed, we have the following basic result.

Proposition 1: *Consider a network of n agents interacting according to an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and consider the optimization problem in eq.* (3)*. If $\mathcal{G}$ is connected then there exists $x_k^* \in \mathbb{R}$ such that $\boldsymbol{x}^*(k) = x_k^* \mathbf{1}$. Moreover:*

i) *If $p=2$ and $\mathcal{X}_{i,k} = \mathbb{R}$, then $x_k^* = \mathrm{avg}(\boldsymbol{u}(k))$;*
ii) *If $p=2$ and $\mathcal{X}_{i,k} = \{x \geq u_i(k)\}$, then $x_k^* = \max(\boldsymbol{u}(k))$;*
iii) *If $p=1$ and $\mathcal{X}_{i,k} = \mathbb{R}$, then $x_k^* = \mathrm{med}(\boldsymbol{u}(k))$.*

Proof: The optimization problem is separable over the single components x_i, which are constrained to be all equal, i.e., $x_i = x_j$ for any $i, j \in \mathcal{V}$ due to the connectedness of graph $\mathcal{G}$. Thus, there exists $x_k^* \in \mathbb{R}$ such that $x_i^*(k) = x_k^*$ for all $i \in \mathcal{V}$. Letting $\tilde{\mathcal{X}}_k = \mathcal{X}_k^1 \cap \cdots \cap \mathcal{X}_k^n$, x_k^* is then solution of the following optimization problem,

$$x_k^* = \operatorname*{argmin}_{x \in \tilde{\mathcal{X}}_k} \left\{ \frac{1}{p} \sum_{i=1}^{n} |x - u_i(k)|^p \right\}. \tag{4}$$

i) If $\mathcal{X}_{i,k} = \mathbb{R}$ then $\tilde{\mathcal{X}}_k = \mathbb{R}$, and if $p = 2$ the solution x_k^* to problem (4) is the average $\mathrm{avg}(\boldsymbol{u})$ (cfr. **Bastianello2022admm**);
ii) If $\mathcal{X}_{i,k} = \{x \geq u_i(k)\}$ then $\tilde{\mathcal{X}}_k = \{x \geq \max(\boldsymbol{u}(k))\}$, and if $p = 2$ the solution x_k^* to problem (4) is the maximum $\max(\boldsymbol{u})$ (cfr. [30]);
iii) If $\mathcal{X}_{i,k} = \mathbb{R}$ then $\tilde{\mathcal{X}}_k = \mathbb{R}$, and if $p = 1$ the solution x_k^* to problem (4) is the median $\mathrm{med}(\boldsymbol{u})$ (cfr. [28]). ■

IV. ADMM PROTOCOLS FOR DYNAMIC CONSENSUS

The implicit updates of the *distributed operator theoretical alternating direction method of multipliers* (DOT-ADMM) applied to a distributed optimization problem in the form of eq. (3) over a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ are given by (cfr. [23, Section III-B] or [22, Section III]),

$$x_i(k) = \operatorname*{argmin}_{x_i \in \mathcal{X}_{i,k}} \left\{ g_{i,k}(x_i) + \frac{\rho \eta_i}{2} x_i^2 - x_i \sum_{j \in \mathcal{N}_i} z_{ij}(k-1) \right\} \tag{5a}$$

$$z_{ij}(k) = (1-\alpha) z_{ij}(k-1) + \alpha (2\rho x_j(k) - z_{ji}(k-1)). \tag{5b}$$

where $g_{i,k}(x_i) = \frac{1}{2}|x_i - u_i(k)|^p$; $x_i \in \mathbb{R}$ with $(i,j) \in \mathcal{V}$ are the primary variables tracking the objective function; $z_{ij} \in \mathbb{R}$ with $(i,j) \in \mathcal{E}$ are auxiliary variables maintained by agent i and sent to agent j; $\alpha \in (0,1)$ and $\rho > 0$ are free parameters. We provide in the following lemmas the explicit updates of the internal variable $\boldsymbol{x}_i(k)$ in eq. (5a) for each objective function of interest.

Lemma 1 (Average): *The explicit DOT-ADMM update in eq.* (5a) *which solves the "dynamic average consensus" problem over a connected network $\mathcal{G}$ is given by*

$$x_i(k) = \frac{u_i(k) + \sum_{j \in \mathcal{N}_i} z_{ij}(k-1)}{1 + \rho \eta_i}.$$

Proof: According to Proposition 1, the dynamic average consensus problem corresponds to the cost functions $g_i(x_i) = \frac{1}{2}(x_i - u_i(k))^2$ without constraints $\mathcal{X}_{i,k} = \mathbb{R}$. Then, the function $f_i(x_i)$ to be minimized in eq. (5a) is quadratic,

$$f_i(x_i) = \frac{1}{2} a x_i^2 - b x_i + c \tag{6}$$

where $a = 1 + \rho \eta_i$ and $b = u_i(k) + \sum_{j \in \mathcal{N}_i} z_{ij}(k-1)$. Since f_i is strongly convex, we can compute its unique minimizer by finding the stationary point of the derivative $\partial f_i = a x_i - b$ that is $x_i = b/a$, which concludes the proof. ■

Remark 1: *The algorithm provided by Lemma 1 is a special case of the algorithm proposed in* **Bastianello2022admm***, which addresses the more complicated scenario of asynchronous, noisy and unreliable communications.*

Lemma 2 (Maximum): *The explicit DOT-ADMM update in eq.* (5a) *which solves the "dynamic maximum consensus" problem over a connected network $\mathcal{G}$ is given by*

$$x_i(k) = \max \left\{ u_i(k), \frac{u_i(k) + \sum_{j \in \mathcal{N}_i} z_{ij}(k-1)}{1 + \rho \eta_i} \right\}.$$

Proof: According to Proposition 1, the dynamic maximum consensus problem corresponds to the cost functions $g_i(x_i) = \frac{1}{2}(x_i - u_i(k))^2$ with constraint sets $\mathcal{X}_{i,k} = \{x \geq u(_i k)\}$. Then, the function $f_i(x_i)$ to be minimized in eq. (5a) is quadratic as in eq. (6) where $a = 1 + \rho \eta_i$ and $b = u_i(k) + \sum_{j \in \mathcal{N}_i} z_{ij}(k-1)$. Due to the constraint set, we derive the KKT conditions are as follows [31, Chapter 5].

$$\text{(stationarity)} \quad a x_i - b - \lambda = 0 \tag{7}$$

$$\text{(complementary slackness)} \quad \lambda (u_i(k) - x_i) = 0 \tag{8}$$

$$\text{(primal feasibility)} \quad u_i(k) - x_i \leq 0 \tag{9}$$

$$\text{(dual feasibility)} \quad \lambda \geq 0. \tag{10}$$

By substituting the value of x_i obtained from eq. (7) into eq. (9) one gets $\lambda \geq a u_i(k) - b$. To ensure that also (8) and (10) are satisfied, it must be $\lambda = \max\{a u_i(k) - b, 0\}$. Substituting this value into eq (7) yields $x_i = \max\{u_i(k), b/a\}$, completing the proof. ■

Lemma 3 (Median): *The explicit DOT-ADMM update in eq.* (5) *which solves the "dynamic median consensus" problem over a connected network $\mathcal{G}$ is given by*

$$x_i(k) = u_i(k) + \max\{\theta_i^- - u_i(k), 0\} + \min\{\theta_i^+ - u_i(k), 0\}$$

where

$$\theta_i^{\pm}(k) = \frac{\sum_{j \in \mathcal{N}_i} z_{ij}(k-1) \pm 1}{\rho \eta_i}.$$

Proof: According to Proposition 1, the dynamic median consensus problem corresponds to the cost functions $g_i(x_i) = |x_i - u_i(k)|$ without constraints $\mathcal{X}_{i,k} = \mathbb{R}$. Then, the update in eq. (5a) can be rewritten as follows,

$$x_i(k) = \mathrm{prox}_{g_{i,k}}^{1/\rho \eta_i} \Big(\frac{1}{\rho \eta_i} \sum_{j \in \mathcal{N}_i} z_{ij}(k-1) \Big).$$

where the proximal operator is defined by $\mathrm{prox}_f^{\lambda}(\boldsymbol{y}) := \mathrm{argmin}_{\boldsymbol{x}} \{ f(\boldsymbol{x}) + \frac{1}{2\lambda} \|\boldsymbol{x} - \boldsymbol{y}\|^2 \}$. Exploiting the composition property, i.e., $\mathrm{prox}_{g_{i,k}}(v) = u_i(k) + \mathrm{prox}_{|\cdot|}(v - u_i(k))$ (see [31, Section 2.2]), and the specific form of the proximal operator associated to the absolute function, i.e., $\mathrm{prox}_{|\cdot|}^{\lambda}(v) = \max\{v - \lambda, 0\} + \min\{v + \lambda, 0\}$ (see [31, Section 6.3.2]), the thesis follows. ■

We next prove that the convergence rate of the ADMM protocols for the estimate of the average, maximum, and median values is R-linear according to Definition 1. More precisely, Theorem 1 proves that the error converges R-linearly to an upper bound that depends on the bound $\sigma \geq 0$ on the reference signals' derivatives as well as on the topological network structure. Consequently, Corollary 1 remarks that for constant reference signals the error converges to zero.

Theorem 1: *Consider a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ executing the protocol in eq. (5) to solve the dynamic consensus problem on the function* $\mathrm{obj} \in \{\mathrm{avg}, \max, \mathrm{med}\}$ *in the case of time-varying reference signals $u_i(k)$ with bounded derivative $\sigma \geq 0$ as in Assumption 1. If the graph $\mathcal{G}$ is connected, the tracking error $e(k) = \|\boldsymbol{x}(k) - \mathrm{obj}(\boldsymbol{u}(k))\mathbf{1}\|$ converges R-linearly to an interval $[0, \varepsilon]$, where $\varepsilon \propto \sigma\sqrt{|\mathcal{E}|}$.*

Corollary 1: *Consider the set-up of Theorem 1 when the reference signals are constant, i.e., $\sigma = 0$. If the graph $\mathcal{G}$ is connected, the tracking error $e(k) = \|\boldsymbol{x}(k) - \mathrm{obj}(\boldsymbol{u})\mathbf{1}\|$ converges R-linearly to zero.*

Proof of Theorem 1: Let $\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{z} \in \mathbb{R}^{|\mathcal{E}|}$ be the vectors stacking $x_i \in \mathbb{R}$ and $z_{ij} \in \mathbb{R}$ with $j \in \mathcal{N}_i$, respectively. From the component-wise DOT-ADMM updates in eq. (5), one can notice that $\boldsymbol{x}(k), \boldsymbol{z}(k)$ are time-varying functions of $\boldsymbol{z}(k-1)$,

$$\boldsymbol{x}(k) = \mathsf{F}_k(\boldsymbol{z}(k-1)),$$
$$\boldsymbol{z}(k) = \mathsf{T}_k^\alpha(\boldsymbol{z}(k-1))$$

where the explicit forms of $\mathsf{F}_k : \mathbb{R}^{|\mathcal{E}|} \to \mathbb{R}^n$ are given component-wise by Lemmas 1-2-3 when the function to be tracked is the average, maximum or median, respectively, and where T_k^α is a convex combination of the identity operator Id and the function $\mathsf{T}_k : \mathbb{R}^{|\mathcal{E}|} \to \mathbb{R}^{|\mathcal{E}|}$, as follows

$$\begin{aligned} \mathsf{T}_k^\alpha &:= (1-\alpha)\mathsf{Id} + \alpha\mathsf{T}_k, \\ \mathsf{T}_k &:= 2\rho P A \mathsf{F}_k - P\mathsf{Id}, \end{aligned} \tag{11}$$

with $P \in \{0,1\}^{|\mathcal{E}|\times|\mathcal{E}|}$ being a permutation matrix swapping $(i,j) \in \mathcal{E}$ with $(j,i) \in \mathcal{E}$ and $A \in \{0,1\}^{|\mathcal{E}|\times n}$ being a block diagonal matrix where each block is the vector $\mathbf{1}_{\eta_i}$. By further defining the *projection operator* over a non-empty, closed and convex set $\mathcal{X}$, by $\mathrm{proj}_{\mathcal{X}}(\boldsymbol{y}) = \mathrm{arginf}_{\boldsymbol{x}\in\mathcal{X}} \|\boldsymbol{x} - \boldsymbol{y}\|^2$, the thesis follows by two intermediate results:

a) There exist $\delta \propto \sigma\sqrt{|\mathcal{E}|}$ such that the distance of $\boldsymbol{z}(k)$ from the set of fixed points of T_k given by
$$d(k) := \left\|\boldsymbol{z}(k) - \mathrm{proj}_{\mathrm{fix}(\mathsf{T}_k)}(\boldsymbol{z}(k))\right\|$$
converges Q-linearly to $[0, \delta]$;

b) There exist $c \geq 0$ such that
$$|e(k) - [0, c\delta]| \leq c|d(k) - [0, \delta]|.$$

Indeed, if claim a) holds then $|d(k) - [0,\delta]|$ converges Q-linearly to zero and, in turn, $e(k)$ converges R-linearly to $[0, \varepsilon]$ where $\varepsilon = c\delta \propto \sigma\sqrt{|\mathcal{E}|}$, completing the proof.

Proof of claim a): Since the updates $\boldsymbol{z}(k) = \mathsf{T}_k^\alpha(\boldsymbol{z}(k-1))$ are the result of the application of the Peaceman-Rachford operator to the dual problem in eq. (3), the operator T_k^α is α-averaged as long as the costs are convex [32, p. 28] as it is the case for the average, the maximum, and the median. By construction, the fixed points of T_k^α coincide with those of T_k, and thus for any fixed point $\hat{\boldsymbol{z}} \in \mathrm{fix}(\mathsf{T}_k)$, α-averagedness of T_k^α implies that

$$\|\mathsf{T}_k^\alpha(\boldsymbol{z}) - \hat{\boldsymbol{z}}\|^2 \leq \|\boldsymbol{z} - \hat{\boldsymbol{z}}\|^2 - \frac{1-\alpha}{\alpha}\|\boldsymbol{z} - \mathsf{T}_k^\alpha(\boldsymbol{z})\|^2. \tag{12}$$

Moreover, since both F_k and T_k are piece-wise affine as previously pointed out, the map T_k^α is piece-wise affine too, and thus also *globally metric subregular* [25, section IV.B], i.e., there is a constant $\gamma > 1/2\alpha$ such that, cfr. [25, eq. (8)],

$$d_{\mathsf{T}_k}(\boldsymbol{z}) := \left\|\boldsymbol{z} - \mathrm{proj}_{\mathrm{fix}(\mathsf{T}_k)}(\boldsymbol{z})\right\| \leq \gamma\|\boldsymbol{z} - \mathsf{T}_k^\alpha(\boldsymbol{z})\|, \tag{13}$$

where $d_{\mathsf{T}_k}(\boldsymbol{z})$ denotes the distance of $\boldsymbol{z}$ from $\mathrm{fix}(\mathsf{T}_k)$. Note that for $\boldsymbol{z} = \boldsymbol{z}(k)$, then $d(k) \equiv d_{\mathsf{T}_k}(\boldsymbol{z}(k))$. The following chain of inequality holds

$$\begin{aligned} d_{\mathsf{T}_k}(\boldsymbol{z}(k))^2 &\overset{(i)}{\leq} \left\|\boldsymbol{z}(k) - \mathrm{proj}_{\mathrm{fix}(\mathsf{T}_k)}(\boldsymbol{z}(k-1))\right\|^2 \\ &\overset{(ii)}{\leq} d_{\mathsf{T}_k}(\boldsymbol{z}(k-1))^2 - \frac{1-\alpha}{\alpha}\|\boldsymbol{z}(k-1) - \boldsymbol{z}(k)\|^2 \\ &\overset{(iii)}{\leq} d_{\mathsf{T}_k}(\boldsymbol{z}(k-1))^2 - \frac{1-\alpha}{\alpha\gamma^2} d_{\mathsf{T}_k}(\boldsymbol{z}(k-1))^2 \end{aligned}$$

where (i) holds by definition of the projection operator; (ii) holds by averagedness in eq. (12); and (iii) holds by global metric subregularity in eq. (13). Thus, we can write

$$d_{\mathsf{T}_k}(\boldsymbol{z}(k)) \leq \mu\, d_{\mathsf{T}_k}(\boldsymbol{z}(k-1)), \quad \mu := \sqrt{1 - \frac{1-\alpha}{\alpha\gamma^2}}, \tag{14}$$

where $\mu \in (0,1)$ for any choice of $\alpha \in (0,1)$ and $\gamma \geq 1/2\alpha$. Eq. (14) means that the application of map T_k^α reduces the distance to its set of fixed points $\mathrm{fix}(\mathsf{T}_k^\alpha) \equiv \mathrm{fix}(\mathsf{T}_k)$. However, the set of fixed points at k is different from that at $k-1$, thus we further need to characterize how the distance $d_{\mathsf{T}_k}(\boldsymbol{z}(k))$ is compared to $d_{\mathsf{T}_{k-1}}(\boldsymbol{z}(k-1))$. By substituting the optimal solution $\boldsymbol{x}^*(k) = \mathsf{F}_k(\boldsymbol{z}(k-1)) = \mathrm{obj}(\boldsymbol{u}(k))\mathbf{1}$ given by Proposition 1 into eq. (11), the set of fixed points at time k results in

$$\mathrm{fix}(\mathsf{T}_k) = \{\boldsymbol{z} \in \mathbb{R}^n \;| M\boldsymbol{z} = \rho\,\mathrm{obj}(\boldsymbol{u}(k))\mathbf{1}\},$$

where $M = (I+P)/2$ is such that[1] $M^2 = M$ and $M^\dagger = M$. Since $\mathrm{fix}(\mathsf{T}_k)$ is an affine set, by [33, section 6.2.2] the projection onto it of any $\boldsymbol{z}$ is given by

$$\begin{aligned} \mathrm{proj}_{\mathrm{fix}(\mathsf{T}_k)}(\boldsymbol{z}) &= \boldsymbol{z} - M^\dagger\left(M\boldsymbol{z} - \rho\,\mathrm{obj}(\boldsymbol{u}(k))\mathbf{1}\right) \\ &= \boldsymbol{z} - \left(M^\dagger M\boldsymbol{z} - \rho\,\mathrm{obj}(\boldsymbol{u}(k))M^\dagger\mathbf{1}\right) \\ &= (I - M)\boldsymbol{z} + \rho\,\mathrm{obj}(\boldsymbol{u}(k))\mathbf{1} \end{aligned} \tag{15}$$

[1]It follows by the facts that P is symmetric $P^\top = P$, row-stochastic $P\mathbf{1} = \mathbf{1}$, and a permutation of order 2, i.e., $P^2 = I$

Thus, the following chain of inequalities holds

$$
\begin{aligned}
d_{\mathsf{T}_k}(\boldsymbol{z}(k)) &\overset{(i)}{\leq} \mu d_{\mathsf{T}_k}(\boldsymbol{z}(k-1)) \\
&= \mu \left\| \boldsymbol{z}(k-1) - \mathrm{proj}_{\mathrm{fix}(\mathsf{T}_k)}(\boldsymbol{z}(k-1)) \right\| \\
&\leq \mu \left\| \boldsymbol{z}(k-1) - \mathrm{proj}_{\mathrm{fix}(\mathsf{T}_k)}(\boldsymbol{z}(k-1)) \pm \mathrm{proj}_{\mathrm{fix}(\mathsf{T}_{k-1})}(\boldsymbol{z}(k-1)) \right\| \\
&\overset{(ii)}{\leq} \mu \left\| \boldsymbol{z}(k-1) - \mathrm{proj}_{\mathrm{fix}(\mathsf{T}_{k-1})}(\boldsymbol{z}(k-1)) \right\| \\
&\quad + \mu \left\| \mathrm{proj}_{\mathrm{fix}(\mathsf{T}_k)}(\boldsymbol{z}(k-1)) - \mathrm{proj}_{\mathrm{fix}(\mathsf{T}_{k-1})}(\boldsymbol{z}(k-1)) \right\| \\
&\overset{(iii)}{\leq} \mu \big(d_{\mathsf{T}_{k-1}}(\boldsymbol{z}(k-1)) + \|\rho(\mathrm{obj}(\boldsymbol{u}(k)) - \mathrm{obj}(\boldsymbol{u}(k-1)))\mathbf{1}\| \big) \\
&\overset{(iv)}{\leq} \mu \big(d_{\mathsf{T}_{k-1}}(\boldsymbol{z}(k-1)) + \rho\sigma\|\mathbf{1}\| \big) \\
&\overset{(v)}{\leq} \mu \Big(d_{\mathsf{T}_{k-1}}(\boldsymbol{z}(k-1)) + \rho\sigma\sqrt{|\mathcal{E}|} \Big)
\end{aligned}
$$

where (i) holds with $\mu \in (0,1)$ by eq. (14); (ii) follows by triangle inequality; (iii) by the specific form of the projection onto an affine set in eq. (15); (iv) by the assumption on the boundedness of the reference signals' derivative; (v) holds since $|\mathcal{E}|$ is the number of components of $\boldsymbol{z}(k)$. Recalling that $d(k) \equiv d_{\mathsf{T}_k}(\boldsymbol{z}(k))$, letting $\tau := \rho\sigma\sqrt{|\mathcal{E}|}$ yields

$$d(k) \leq \mu(d(k-1) + \tau). \tag{16}$$

Moreover, iterating eq. (16) over k yields

$$d(k) \leq \mu^k d(0) + \tau\mu \sum_{i=0}^{k-1} \mu^i \leq \mu^k d(0) + \frac{\tau\mu}{1-\mu}$$

where the last inequality follows by the limit for $k \to \infty$ of the geometric series with $\mu \in (0,1)$. Thus, it holds

$$\lim_{k\to\infty} d(k) \leq \frac{\tau\mu}{1-\mu} =: \delta \tag{17}$$

i.e., $d(k)$ converges to the interval $[0, \delta]$, and, moreover,

$$\exists k^* \in \mathbb{R} \cup \{\infty\} : \quad \begin{cases} d(k) > \delta & \text{if } k < k^* \\ d(k) \in [0,\delta] & \text{otherwise} \end{cases}. \tag{18}$$

We now show that $d(k)$ converges Q-linearly to $[0, \delta]$ for $k \in [0, k^*)$, such that $d(k) > \delta$ and $d(k-1) > \delta$, as follows

$$\frac{|d(k) - [0,\delta]|}{|d(k-1) - [0,\delta]|} \leq \frac{\mu d(k-1) + \mu\tau - \delta}{d(k-1) - \delta} \leq \frac{\mu(d(k-1) - \delta)}{d(k-1) - \delta} = \mu.$$

Proof of claim b): Since $\boldsymbol{x}(k)$ is an internal variable of the update of $\boldsymbol{z}(k)$, then one can show that the error $e(k)$ can be upper-bounded as follows, cfr. [23, eq. (18)],

$$e(k) \leq c d(k), \qquad \text{for some } c > 0.$$

Due to eq. (17), in the limit of $k \to \infty$ it holds

$$\lim_{k\to\infty} e(k) \leq \lim_{k\to\infty} c d(k) \leq c\delta,$$

i.e., $e(k)$ converges to the interval $[0, c\delta]$, and, in turn,

$$|e(k) - [0, c\delta]| \leq c|d(k) - [0,\delta]|, \qquad k \in [0, k^*).$$

V. Numerical simulations

A. Linear convergence and robustness to re-initialization

We substantiate the linear convergence rate and the robustness to re-initialization by simulating a network of $n = 5$ agents interacting according to a line graph when the function to be tracked varies with the maximum rate. Agents' states and reference signals are initialized as follows

$$\boldsymbol{x}(0) = [0,\ 0.5,\ 1\ , 1.5,\ 2]^\mathsf{T}, \qquad \boldsymbol{u}(0) = [0,\ 0,\ 0,\ 2,\ 2]^\mathsf{T}.$$

In order to facilitate the interpretation of this simulation, let us consider the simple yet not trivial scenario where Assumption 1 holds with maximum derivative $\sigma = 0.01$ and where the reference signals are time-varying according to

$$\boldsymbol{u}(k+1) = \begin{cases} \boldsymbol{u}(k) + \sigma & \text{if } k \in (0, 150] \\ \boldsymbol{u}(k) & \text{if } k \in (150, 300] \\ \boldsymbol{u}(k) - \sigma & \text{if } k \in (300, 600] \end{cases}.$$

Note that more challenging behavior of the reference signals can be addressed, as shown next in Section V-B. We also simulate the unexpected disconnection of the 5-th agent from the network at time $k = 450$. Thus, from $k \geq 450$ the network consists of only 4 agents and the function to be tracked is affected by an abrupt change.

Fig. 1 shows the results of the simulation when the function to be tracked is either the average, the maximum, or the median, i.e., $\mathrm{obj} \in \{\mathrm{avg}, \max, \mathrm{med}\}$. One can verify that the agents recover the tracking with linear convergence every time the objective function changes behavior, i.e., at $k = \{150, 300, 450\}$, thus corroborating the results in Theorem 1 and Corollary 1. We note that at $k = 450$ the discontinuous change of the objective function due to the disconnection of agent 5 does not disrupt the function of the algorithm, which is able to recover the tracking with linear convergence without requiring any re-initialization process.

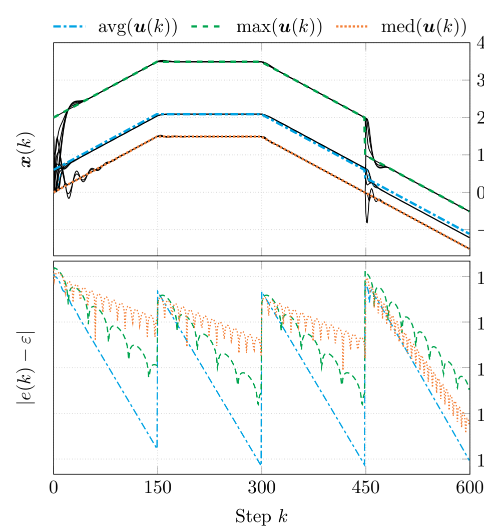
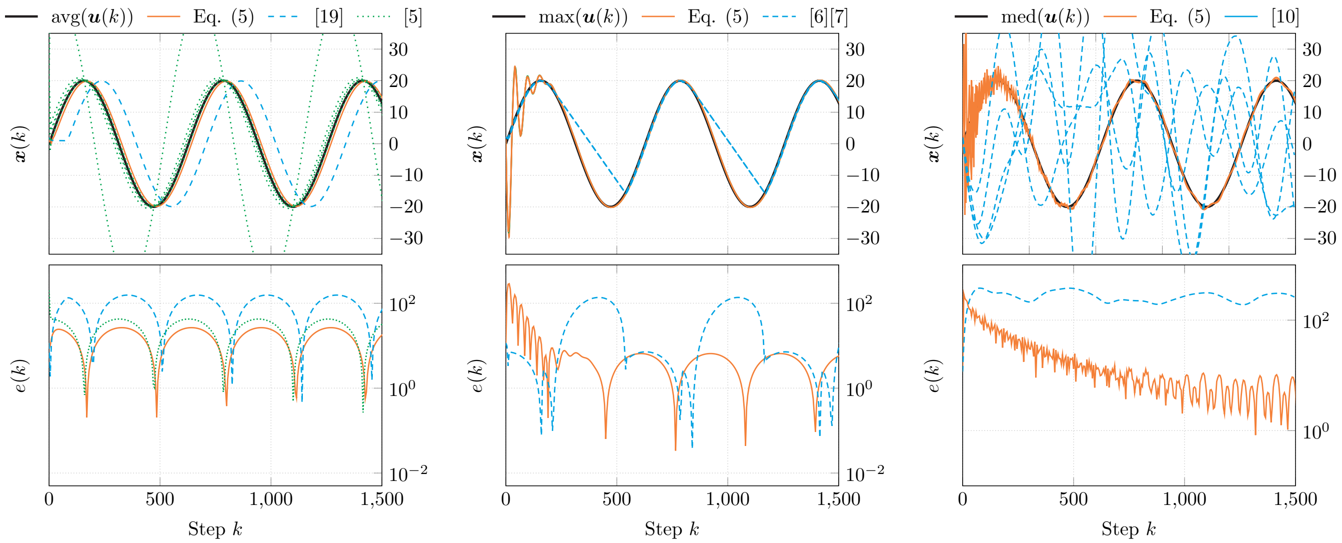


Fig. 1: Evolution of a network, with $n = 5$ agents in a line configuration, running the protocol in eq. (5) designed with $\alpha = 0.5$ and $\rho = 2$. The parameter ε denotes the empirical bound on the tracking error $e(k)$ characterized in Theorem 1.



(a) The protocol in eq. (5) is designed with $\alpha = 0.8$ and $\rho = 2$; the MSDAC protocol in [19] is designed with $m = 40$ stages, $\alpha = 0.49$ and $\varepsilon = 0.027$; the protocol in [5, eq. (31)] is designed as in [5, Table 3].

(b) The protocol in eq. (5) is designed with $\alpha = 0.95$ and $\rho = 10$; the STDMC protocol in [6], [7] is designed with $\alpha^{\text{MAX}} = 0.1$ and $\alpha^{\text{MIN}} = 10^{-8}$.

(c) The protocol in eq. (5) is designed with $\alpha = 0.95$ and $\rho = 0.01$; the protocol in [10] is designed with $\alpha = 2$, $\beta = 0.001$, $\gamma = 0.0005$, and $\kappa = 0.1$.

Fig. 2: Comparison of the protocol in eq. (5) (orange) with the state-of-the-art (blue) on a network with $n = 101$ agents.

B. Comparison with the state-of-the-art

We compare our protocols to those proposed by Van Scoy *et al.* in [5], [34], by Franceschelli *et al.* in [19], by Deplano *et al.* in [6], [7] and by Vasiljevic *et al.* in [10] by considering a network of $n = 101$ agents interacting according to a random topology. The agents' states are initialized uniformly within the interval $[0, 2]$, and the objective function to be tracked is given by

$$\text{obj}(\boldsymbol{u}(k)) = 20 \sin\left(\frac{k}{100}\right). \tag{19}$$

This scenario is similar to simulations in [19, Fig. 3], [7, Fig. 4], [10, Fig. 10], thus making the comparison fair.

Dynamic average consensus. Fig. 2a shows the results of the comparative simulation with the MSDAC protocol proposed by Franceschelli *et al.* in in [19] and the protocol proposed by Van Scoy in [34], which has been included in the recent review [5] due to its robustness to initial conditions and its accelerated convergence time. In order to get a reasonable tracking error, we have designed the MSDAC protocol with a large number of stages equal to $m = 40$, thus yielding to a delayed tracking of the time-varying average of about 70 time steps. A similar effect arises also with the protocol in eq. (5), which is, however, much smaller. The design of the protocol in [34] has been carried out according to [5, Table 3], which optimizes the asymptotic convergence rate over the set of connected undirected graphs. Nevertheless, Fig. 2a reveals that all protocols have similar convergence rates. Moreover, the protocol in [34] does not guarantee that all agents achieve a good estimation of the time-varying average value, in contrast to the proposed protocol.

Dynamic max-consensus. To our knowledge, the only protocol in the current literature to solve the dynamic max-consensus problem in discrete-time multi-agent systems has been developed by Deplano *et al.* in [7], then extended in [6]. Fig. 2b shows the results of the comparative simulation with the STDMC Protocol in [6], [7] in the case the agents do not know a correct upper bound to the derivative of the reference signals. This scenario emphasizes the disadvantage of the STDMC Protocol, i.e., the requirement that the bound σ on the reference signals' derivative is known locally by the agents. Indeed, every time the reference signal decreases at a rate greater than expected, the track is lost by STDMC Protocol, while this is not the case for our protocol in eq. (5), which maintains the tracking all the time.

Dynamic median consensus. Fig. 2c shows the results of the comparative simulation with the protocol proposed by Vasiljevic *et. al* in [10], which fails in the proposed scenario, while the proposed protocol succeeds. Even though we have exhaustively tried all combinations of the tuning parameters for the protocol in [10], it never worked. In our opinion, this is because the authors of [10] have considered reference signals with very similar behavior, i.e., they are all equal up to a constant factor (see Fig. 4 in [10]). This scenario is very similar to that considered in [29], which is, however, a much simpler set-up than the one considered in this manuscript. Indeed, in our simulations, the inputs have all different behaviors, which may be the reason for the failure of the protocol in [10]. This proves the protocol proposed in this work for the dynamic tracking of the median value is the first in the current literature which can actually deal with heterogeneous time-varying reference signals.

VI. Conclusions

A unified analysis technique is employed to prove the linear convergence of original local interaction protocols that solve the dynamic consensus problem on the average, maximum, and median functions. The central feature of this technique is the use of the metric subregularity property [25, section IV.B] which is inherent in the proposed protocols due to their piece-wise affinity. Future research will focus on characterizing the linear convergence rate μ in eq. (14), which is closely correlated to the constant γ of metric subregularity. This would allow for the exploration of optimal parameter designs in terms of convergence time and tracking error.

References

[1] M. Zhu and S. Martínez, "Discrete–time dynamic average consensus," *Automatica*, vol. 46, no. 2, pp. 322–329, 2010.

[2] J. Cortés, "Distributed algorithms for reaching consensus on general functions," *Automatica*, vol. 44, no. 3, pp. 726–737, 2008.

[3] R. A. Freeman, P. Yang, and K. M. Lynch, "Stability and convergence properties of dynamic average consensus estimators," in *Proceedings of the IEEE Conference on Decision and Control*, 2006, pp. 398–403.

[4] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Transactions on Automatic Control*, vol. 50, no. 2, pp. 169–182, 2005.

[5] S. S. Kia, B. Van Scoy, J. Cortes, R. A. Freeman, K. M. Lynch, and S. Martinez, "Tutorial on dynamic average consensus: The problem, its applications, and the algorithms," *IEEE Control Systems*, vol. 39, no. 3, pp. 40–72, 2019.

[6] D. Deplano, M. Franceschelli, and A. Giua, "Dynamic max–consensus with local self–tuning," *IFAC–PapersOnLine*, vol. 55, no. 13, pp. 127–132, 2022.

[7] D. Deplano, M. Franceschelli, and A. Giua, "Dynamic min and max consensus and size estimation of anonymous multiagent networks," *IEEE Transactions on Automatic Control*, vol. 68, no. 1, pp. 202–213, 2023.

[8] M. Lippi, A. Furchi, A. Marino, and A. Gasparri, "An adaptive distributed protocol for finite–time infimum or supremum dynamic consensus," *IEEE Control Systems Letters*, vol. 7, pp. 401–406, 2023.

[9] Z. A. Z. Sanai Dashti, C. Seatzu, and M. Franceschelli, "Dynamic consensus on the median value in open multi–agent systems," in *Proceedings of the IEEE Conference on Decision and Control*, 2019, pp. 3691–3697.

[10] G. Vasiljevic, T. Petrovic, B. Arbanas, and S. Bogdan, "Dynamic median consensus for marine multi–robot systems using acoustic communication," *IEEE Robotics and Automation Letters*, vol. 5, no. 4, pp. 5299–5306, 2020.

[11] D. Varagnolo, G. Pillonetto, and L. Schenato, "Distributed cardinality estimation in anonymous networks," *IEEE Transactions on Automatic Control*, vol. 59, no. 3, pp. 645–659, 2014.

[12] D. Deplano, M. Franceschelli, and A. Giua, "Distributed tracking of graph parameters in anonymous networks with time–varying topology," in *Proceedings of the IEEE Conference on Decision and Control*, vol. 2021–December, 2021, pp. 6258–6263.

[13] E. Montijano, G. Oliva, and A. Gasparri, "Distributed estimation and control of node centrality in undirected asymmetric networks," *IEEE Transactions on Automatic Control*, vol. 66, no. 5, pp. 2304–2311, 2021.

[14] Y. Shang, "Resilient consensus in multi–agent systems with state constraints," *Automatica*, vol. 122, p. 109 288, 2020.

[15] L. Sheng, W. Gu, G. Cao, and X. Chen, "A distributed detection mechanism and attack–resilient strategy for secure voltage control of ac microgrids," *CSEE Journal of Power and Energy Systems*, pp. 1–10, 2022.

[16] E. Nuno, A. Loria, and E. Panteley, "Leaderless consensus formation control of cooperative multi–agent vehicles without velocity measurements," *IEEE Control Systems Letters*, vol. 6, pp. 902–907, 2022.

[17] C. Chen, Y. Han, S. Zhu, and Z. Zeng, "Distributed fixed–time tracking and containment control for second–order multi–agent systems: A nonsingular sliding–mode control approach," English, *IEEE Transactions on Network Science and Engineering*, pp. 1–11, 2022.

[18] M. Santilli, M. Franceschelli, and A. Gasparri, "Dynamic resilient containment control in multirobot systems," *IEEE Transactions on Robotics*, vol. 38, no. 1, pp. 57–70, 2022.

[19] M. Franceschelli and A. Gasparri, "Multi-stage discrete time and randomized dynamic average consensus," *Automatica*, vol. 99, pp. 69–81, 2019.

[20] E. Dall'Anese, A. Simonetto, S. Becker, and L. Madden, "Optimization and learning with information streams: Time–varying algorithms and applications," *IEEE Signal Processing Magazine*, vol. 37, no. 3, pp. 71–83, 2020.

[21] A. Simonetto, E. Dall'Anese, S. Paternain, G. Leus, and G. B. Giannakis, "Time–varying convex optimization: Time–structured algorithms and applications," *Proceedings of the IEEE*, vol. 108, no. 11, pp. 2032–2048, 2020.

[22] N. Bastianello, D. Deplano, M. Franceschelli, and K. H. Johansson, "Online distributed learning over random networks," *arXiv preprint arxiv.org/abs/2309.00520*, 2023.

[23] N. Bastianello, R. Carli, L. Schenato, and M. Todescato, "Asynchronous distributed optimization over lossy networks via relaxed ADMM: Stability and linear convergence," *IEEE Transactions on Automatic Control*, vol. 66, no. 6, pp. 2620–2635, 2021.

[24] S. Boyd, N. Parikh, E. Chu, B. Peleato, J. Eckstein, *et al.*, "Distributed optimization and statistical learning via the alternating direction method of multipliers," *Foundations and Trends® in Machine learning*, vol. 3, no. 1, pp. 1–122, 2011.

[25] A. Themelis and P. Patrinos, "Supermann: A superlinearly convergent algorithm for finding fixed points of nonexpansive operators," English, *IEEE Transactions on Automatic Control*, vol. 64, no. 12, pp. 4875–4890, 2019.

[26] E. Montijano, J. I. Montijano, C. Sagüés, and S. Martínez, "Robust discrete time dynamic average consensus," *Automatica*, vol. 50, no. 12, pp. 3131–3138, 2014.

[27] D. Deplano, M. Franceschelli, and A. Giua, "Discrete–time dynamic consensus on the max value," in *15th European Workshop on Advanced Control and Diagnosis (ACD 2019)*, E. Zattoni, S. Simani, and G. Conte, Eds., Cham: Springer International Publishing, 2022.

[28] M. Franceschelli, A. Giua, and A. Pisano, "Finite-time consensus on the median value with robustness properties," *IEEE Transactions on Automatic Control*, vol. 62, no. 4, pp. 1652–1667, 2016.

[29] S. Yu, Y. Chen, and S. Kar, "Dynamic median consensus over random networks," in *Proceedings of the IEEE Conference on Decision and Control*, vol. 2021, 2021, pp. 5695–5702.

[30] N. K. D. Venkategowda and S. Werner, "Privacy–preserving distributed maximum consensus," English, *IEEE Signal Processing Letters*, vol. 27, pp. 1839–1843, 2020.

[31] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.

[32] E. K. Ryu and S. Boyd, "Primer on monotone operator methods," *Applied Comput. Math*, vol. 15, no. 1, pp. 3–43, 2016.

[33] N. Parikh and S. Boyd, "Proximal algorithms," *Foundations and Trends® in Optimization*, vol. 1, no. 3, pp. 127–239, 2014.

[34] B. V. Scoy, R. A. Freeman, and K. M. Lynch, "Design of robust dynamic average consensus estimators," *2015 54th IEEE Conference on Decision and Control (CDC)*, pp. 6269–6275, 2015.