A unified approach to solve the dynamic consensus on the average, maximum, and median values with linear convergence

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Outline

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Scenarios

Peer-to-Peer Networks

Wireless Sensor Networks

Multi-Robot Systems

Problem set-up

Undirected network → $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Set of agents → $\mathcal{V} = \{1, \dots, n\}$

Set of interactions → $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$

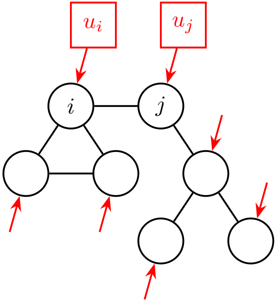
State of agent i → $x_i \in \mathbb{R}$

Reference signal of agent i → $u_i \in \mathbb{R}$

Neighbors of agent i → $\mathcal{N}_i = \{j \mid (i,j) \in \mathcal{E}\}$

Number of neighbors of agent i → $\eta_i = |\mathcal{N}_i| \in \mathbb{R}$

Framework → Discrete-time $k \in \mathbb{N}$



$$x_i(k) = f_i\left(u_i(k), x_i(k-1), x_j(k-1) : j \in \mathcal{N}_i\right), \quad i \in \mathcal{V} \tag{1}$$

Objective

The agents must cooperate to track an objective function $\text{obj}(u(k)) \in \mathbb{R}$ of the reference signals. We focus on the average $\text{avg}(u(k))$, maximum $\max(u(k))$, and median $\text{med}(u(k))$ functions.

Literature

Definition: Dynamic consensus problem

Design the local interaction rules f_i such that the agents' state x_i converges to a scalar function $\mathrm{obj} : \mathbb{R}^n \to \mathbb{R}$ of the reference signals $u_1, \ldots, u_n$, i.e., there exists $\varepsilon \geq 0$ such that

$$\|x_i(k) - \mathrm{obj}(u_1(k), \ldots, u_n(k))\| \leq \varepsilon, \qquad k \geq k^*, \quad i \in \mathcal{V}, \tag{2}$$

The **average** (*sum of values of a data set divided by number of values*):

- Spanos, Olfati-Saber, and Murray, "Dynamic consensus on mobile networks", in *IFAC World Congr.* (2005)
- Freeman, Yang, and Lynch, "Stability and convergence properties of dynamic average consensus estimators", in *IEEE 45th Conf. on Dec. and Control (2006)*
- Zhu and Martinez, "Discrete-time dynamic average consensus", in *Automatica* (2010)
- Chen, Cao and Ren, "Distributed average tracking of multiple time-varying reference signals with bounded derivatives", in *IEEE Trans. Autom. Control* (2012).
- Kia, Cortés, and Martinez "Dynamic average consensus under limited control authority and privacy requirements", in *Int. Journal of Robust and Nonlin. Control* (2015)
- Scoy, Freeman, and Lynch, "A fast robust nonlinear dynamic average consensus estimator in discrete time", in *5th IFAC NecSys* (2015)
- Franceschelli, and Gasparri, "Multi-stage discrete time and randomized dynamic average consensus", in *Automatica* (2019)
- George and Freeman, "Robust dynamic average consensus algorithms", in *IEEE Trans. Autom. Control* (2019)
- Montijano E. and J.I., Sagues, and Martinez, "Robust discrete time dynamic average consensus", in *IEEE Trans. Autom. Control* (2019)
- Kia, Scoy, Cortés, Freeman, Lynch and Martinez, "Tutorial on dynamic average consensus: The problem, its applications, and the algorithms", in *IEEE Control Systems Magazine* (2019).

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$$\|x_i(k) - \text{obj}(u_1(k), \ldots, u_n(k))\| \leq \varepsilon, \qquad k \geq k^*, \quad i \in \mathcal{V}, \tag{2}$$

The **maximum** (*highest value of a data set*).

- Deplano, Franceschelli, Giua, "Dynamic max-consensus with local self-tuning", in *IFAC-PapersOnLine (NecSys)*, (2022)
- Deplano, Franceschelli, Giua, "Discrete-time Dynamic consensus on the max value", in *15th European Workshop on Advanced Control and Diagnosis*, Springer (2021)
- Deplano, Franceschelli, and Giua, "Dynamic min and max consensus and size estimation of anonymous multi-agent networks", in *IEEE Trans. Autom. Control* (2021).
- Sen, Sahoo, and Slingh, "Global max-tracking of multiple time-varying signals using a distributed protocol", in *IEEE Control and Sys. Lett.* (2022)

Literature

Definition: Dynamic consensus problem

Design the local interaction rules f_i such that the agents' state x_i converges to a scalar function $\mathrm{obj}:\mathbb{R}^n\to\mathbb{R}$ of the reference signals $u_1,\dots,u_n$, i.e., there exists $\varepsilon\geq 0$ such that

$$\|x_i(k)-\mathrm{obj}(u_1(k),\dots,u_n(k))\|\leq\varepsilon,\qquad k\geq k^*,\quad i\in\mathcal{V},\tag{2}$$

The **median** (*middle value separating the greater and lesser halves of a data set*):

- Sanai Dashti, Seatzu, and Franceschelli, "Dynamic consensus on the median value in open multi-agent systems", in *IEEE 58th Conf. on Dec. and Control* (2019).
- Vasiljevic, Petrovic, Arbanas, and Bogdan, "Dynamic median consensus for marine multi-robot systems using acoustic communication", in *IEEE Robot. and Autom. Lett.* (2020).
- Yu, Chen and Kar, "Dynamic median consensus over random networks", in *IEEE 60th Conf. on Dec. and Control* (2021).

Outline

Dynamic consensus as a time-varying optimization problem

The dynamic consensus problems on average, maximum, and median functions can be recast as distributed time-varying optimization problems of the following type

$$x^*(k) = \underset{x_1,\dots,x_n}{\operatorname{argmin}} \sum_{i=1}^{n} \frac{1}{p}|x_i - u_i(k)|^p \qquad (3)$$
$$\text{s.t.} \quad x_i = x_j \quad \forall\, (i,j) \in \mathcal{E}$$
$$x_i \in \mathcal{X}_{i,k} \quad \forall\, i \in \mathcal{V}.$$

If $\mathcal{G}$ is connected then there exists $x_k^* \in \mathbb{R}$ such that $x^*(k) = x_k^* \mathbf{1}$. Moreover:

i) If $p = 2$ and $\mathcal{X}_{i,k} = \mathbb{R}$, then $x_k^* = \operatorname{avg}(u(k))$;

ii) If $p = 2$ and $\mathcal{X}_{i,k} = \{x \geq u_i(k)\}$, then $x_k^* = \max(u(k))$;

iii) If $p = 1$ and $\mathcal{X}_{i,k} = \mathbb{R}$, then $x_k^* = \operatorname{med}(u(k))$.

DOT-ADMM: DISTRIBUTED OPERATOR THEORETICAL (DOT) ADMM ALGORITHM

(Input): Relaxation parameter $\alpha \in (0,1)$; penalty parameter ρ
(Initialization): $x_i(0), z_{ij}(0) \in \mathbb{R}$ for $i \in \mathcal{V}$ and $j \in \mathcal{N}_i$
(Output): Each node $i \in \mathcal{V}$ outputs the approximated solution $x_i(k)$ to the optimization problem
(Execution): for $k = 1, 2, 3, \dots$ **each node** i **does**

1) Update the local cost $f_{i,k}$ and update the state variable

$$x_i(k) = \operatorname*{argmin}_{x_i \in \mathcal{X}_{i,k}} \left\{ \frac{1}{p} |x_i - u_i(k)|^p + \frac{\rho \eta_i}{2} x_i^2 - x_i \sum_{j \in \mathcal{N}_i} z_{ij}(k-1) \right\} \tag{4}$$

2) Transmit a packet $y_{i \to j}$ to each neighbor $j \in \mathcal{N}$,

$$y_{i \to j}(k) = 2\rho x_i(k) - z_{ij}(k-1);$$

3) For each packet $y_{j \to i}$ received by a neighbor $j \in \mathcal{N}_i$ update the auxiliary variable

$$z_{ij}(k) = (1-\alpha) z_{ij}(k-1) + \alpha y_{j \to i}(k).$$

N. Bastianello, D. Deplano, M. Franceschelli, K.H. Johansson, "Online distributed learning over random networks", Transactions on Automatic Control (under review)

Main results

Theorem 1: Explicit updates for average, maximum and median consensus

The explicit DOT-ADMM updates in eq. (4) for solving the "dynamic consensus" problem, under Assumption 1 and over a connected network $\mathcal{G}$, are given by:

- Dynamic average consensus:

$$x_i(k) = \frac{u_i(k) + \sum_{j \in \mathcal{N}_i} z_{ij}(k-1)}{1 + \rho \eta_i}.$$

- Dynamic maximum consensus:

$$x_i(k) = \max \left\{ u_i(k), \frac{u_i(k) + \sum_{j \in \mathcal{N}_i} z_{ij}(k-1)}{1 + \rho \eta_i} \right\}.$$

- Dynamic median consensus:

$$x_i(k) = u_i(k) + \max\{\theta_i^- - u_i(k), 0\} + \min\{\theta_i^+ - u_i(k), 0\}$$

  where

$$\theta_i^{\pm}(k) = \frac{\sum_{j \in \mathcal{N}_i} z_{ij}(k-1) \pm 1}{\rho \eta_i}.$$

Main results

Assumption 1

The variation of the reference signals $u_i(k)$ are bounded a constant $\Pi \geq 0$, i.e., for $k \geq 0$ it holds

$$\Delta u_i(k) = |u_i(k) - u_i(k-1)| \leq \sigma$$

Theorem 2: Linear convergence and bounded tracking error

Consider a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ executing the DOT-ADMM Algorithm to solve the dynamic average, maximum, and median consensus problem in the case of time-varying reference signals $u_i(k)$ with bounded derivative $\sigma \geq 0$ as in Assumption 1. If the graph $\mathcal{G}$ is connected:

- The tracking error $e(k) = \|x(k) - \mathrm{obj}(u(k))\mathbf{1}\|$ converges R-linearly to an interval $\propto [0, \sigma]$.

Outline

Linear convergence and robustness to re-initialization

- Network of $n = 5$ agents with:

$$\alpha = 0.5, \quad \rho = 2, \quad \sigma = 0.01$$

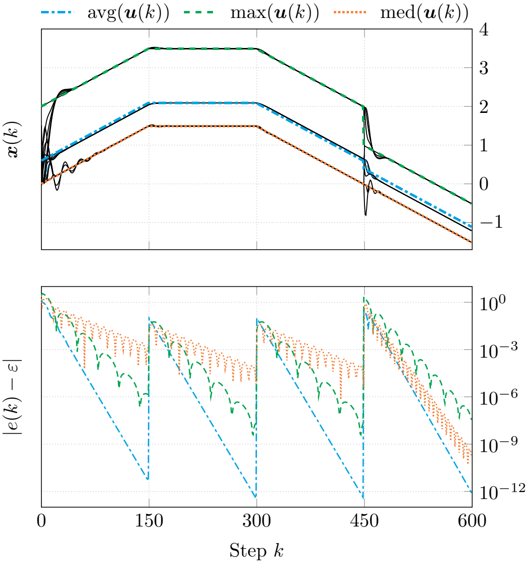
- Agents' states and reference signals are initialized as

$$x(0) = [0,\ 0.5,\ 1\ ,1.5,\ 2]^\top$$
$$u(0) = [0,\ 0,\ 0,\ 2,\ 2]^\top.$$

- Reference signals vary according to

$$u(k+1) = \begin{cases} u(k) + \sigma & \text{if } k \in (0, 150] \\ u(k) & \text{if } k \in (150, 300] \\ u(k) - \sigma & \text{if } k \in (300, 600] \end{cases}$$

- Unexpected disconnection of an agent at At step $k = 450$: for $k \geq 450$ the network consists of only 4 agents



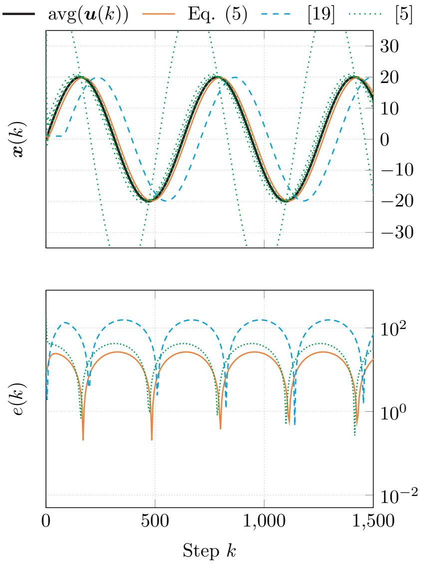
Comparison with the state-of-the-art: dynamic average consensus

Comparison with:

- [5] S. S. Kia, B. Van Scoy, J. Cortes, R. A. Freeman, K. M. Lynch, and S. Martinez, "Tutorial on dynamic average consensus: The problem, its applications, and the algorithms", IEEE Control Systems, 2019.
- [19] M. Franceschelli and A. Gasparri, "Multi-stage discrete time and randomized dynamic average consensus", Automatica, 2019.

Conclusions:

- The protocol in [19] is affected by a "delay" of about 70 time steps, much larger than the proposed protocol;
- The protocol in [5] does not guarantee that all agents achieve a good estimation of the time-varying average value, in contrast to the proposed protocol;
- Similar convergence rates.



Comparison with the state-of-the-art: dynamic maximum consensus

Comparison with:

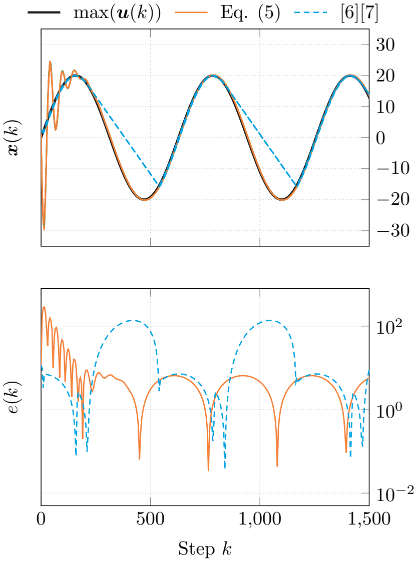
- [6] D. Deplano, M. Franceschelli, and A. Giua, "Dynamic max–consensus with local self–tuning", IFAC–PapersOnLine (NecSys), 2022.
- [7] D. Deplano, M. Franceschelli, and A. Giua, "Dynamic min and max consensus and size estimation of anonymous multiagent networks", IEEE Transactions on Automatic Control, 2023.

Set-up:

- The agents do not know the correct upper bound to the derivative of the reference signals.

Conclusion:

- Every time the reference signal decreases at a rate greater than expected, the track is lost by the protocol in [6][7], while the proposed protocol maintains the tracking all the time



Comparison with the state-of-the-art: dynamic median consensus

Comparison with:

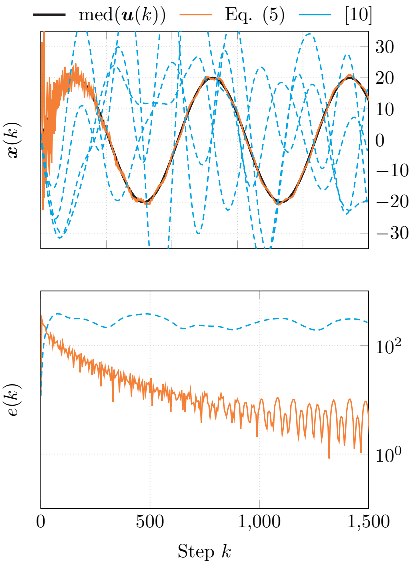
- [10] G. Vasiljevic, T. Petrovic, B. Arbanas, and S. Bogdan, "Dynamic median consensus for marine multi–robot systems using acoustic communication", IEEE Robotics and Automation Letters, 2020.

Set-up:

- The reference signals have heterogeneous behavior.

Conclusion:

- The protocol in [10] fails in converging to the median value, thus the proposed protocol is the first in the current literature to solve this problem.



Outline

Conclusions and future directions

Contribution 1

Three novel protocols to solve the dynamic consensus on the average, maximum, and median functions are proposed, with improved performance with respect to the state-of-the-art:

- **Better trade-off** between convergence rate and tracking error for the dynamic average consensus;
- **Higher robustness** to unexpected spikes in the inputs' variation for the dynamic max consensus;
- **The first and only protocol** that solves the dynamic median consensus with heterogeneous inputs.

Contribution 2

The protocols are derived within a unified framework by exploiting the newly proposed DOT-ADMM*, and have shown to have the following properties:

- **Linear convergence rate** for a class of (not necessarily strong) convex problems;
- **Robustness to re-initialization**;

*N. Bastianello, D. Deplano, M. Franceschelli, K.H. Johansson, "Online distributed learning over random networks", Transactions on Automatic Control (under review)

Conclusions and future directions

Future directions

We will investigate the following properties of the DOT-ADMM for the specific protocols we have presented:

- **Robustness to asynchronous and noisy communications**;
- **Robustness to unreliable communications**.

Moreover, we aim at:

- Formally characterize the bound on the tracking error;
- Extend their applicability to open networks where agents may join and leave the network over time.

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Thank you for your attention!

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