Dynamic Max-Consensus with Local Self-Tuning

Diego Deplano*, **Mauro Franceschelli***, **Alessandro Giua***

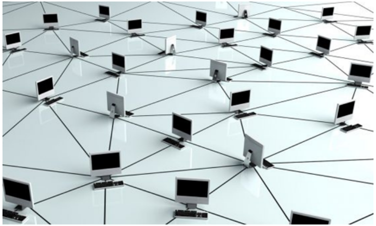
*Department of Electrical and Electronic Engineering, University of Cagliari, Italy

IFAC Conference on Networked Systems (NecSys), 5-7 July 2022

Outline

Outline

Scenarios



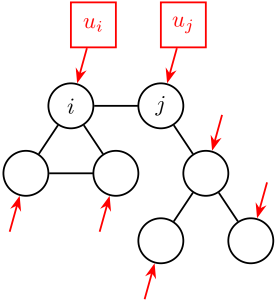
Peer-to-Peer Networks

Wireless Sensor Networks

Multi-Robot Systems

Problem set-up

Undirected network	$\rightarrow \mathcal{G} = (\mathcal{V}, \mathcal{E})$
Set of agents	$\rightarrow \mathcal{V} = \{1, \ldots, n\}$
Set of interactions	$\rightarrow \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$
State of agent i	$\rightarrow x_i \in \mathbb{R}$
Reference signal of agent i	$\rightarrow u_i \in \mathbb{R}$
Neighbors of agent i	$\rightarrow \mathcal{N}_i = \{j \mid (i,j) \in \mathcal{E}\}$
Framework	$\rightarrow$ Discrete-time $k \in \mathbb{N}$



$$x_i(k) = f_i\left(u_i(k),\, x_i(k-1),\, x_j(k-1) : j \in \mathcal{N}_i\right), \quad i \in \mathcal{V} \tag{1}$$

Objective

The agents must cooperate to track the time-varying **maximum value** of the reference signals,

$$\overline{u}(k) = \max\{u_1(k), \ldots, u_n(k)\}.$$

Literature

Definition: Dynamic consensus problem

Design the local interaction rules f_i such that the agents' state x_i converges to a scalar function $g : \mathbb{R}^n \to \mathbb{R}$ of the reference signals $u_1, \ldots, u_n$, i.e., there exists $\varepsilon \geq 0$ such that

$$\|x_i(k) - g(u_1(k), \ldots, u_n(k))\| \leq \varepsilon, \qquad k \geq k^*, \quad i \in \mathcal{V},$$

The **average** (*sum of values of a data set divided by number of values*):

- Spanos, Olfati-Saber, and Murray, "Dynamic consensus on mobile networks", in *IFAC World Congr.* (2005)
- Freeman, Yang, and Lynch, "Stability and convergence properties of dynamic average consensus estimators", in *IEEE 45th Conf. on Dec. and Control (2006)*
- Zhu and Martinez, "Discrete-time dynamic average consensus", in *Automatica* (2010)
- Chen, Cao and Ren, "Distributed average tracking of multiple time-varying reference signals with bounded derivatives", in *IEEE Trans. Autom. Control* (2012).
- Kia, Cortés, and Martinez "Dynamic average consensus under limited control authority and privacy requirements", in *Int. Journal of Robust and Nonlin. Control* (2015)
- Scoy, Freeman, and Lynch, "A fast robust nonlinear dynamic average consensus estimator in discrete time", in *5th IFAC NecSys* (2015)
- Franceschelli, and Gasparri, "Multi-stage discrete time and randomized dynamic average consensus", in *Automatica* (2019)
- George and Freeman, "Robust dynamic average consensus algorithms", in *IEEE Trans. Autom. Control* (2019)
- Montijano E. and J.I., Sagues, and Martinez, "Robust discrete time dynamic average consensus", in *IEEE Trans. Autom. Control* (2019)
- Kia, Scoy, Cortés, Freeman, Lynch and Martinez, "Tutorial on dynamic average consensus: The problem, its applications, and the algorithms", in *IEEE Control Systems Magazine* (2019).

Literature

Definition: Dynamic consensus problem

Design the local interaction rules f_i such that the agents' state x_i converges to a scalar function $g : \mathbb{R}^n \to \mathbb{R}$ of the reference signals $u_1, \ldots, u_n$, i.e., there exists $\varepsilon \geq 0$ such that

$$\|x_i(k) - g(u_1(k), \ldots, u_n(k))\| \leq \varepsilon, \qquad k \geq k^*, \quad i \in \mathcal{V},$$

The **median** (*middle value separating the greater and lesser halves of a data set*):

- Sanai Dashti, Seatzu, and Franceschelli, "Dynamic consensus on the median value in open multi-agent systems", in *IEEE 58th Conf. on Dec. and Control* (2019).
- Vasiljevic, Petrovic, Arbanas, and Bogdan, "Dynamic median consensus for marine multi-robot systems using acoustic communication", in *IEEE Robot. and Autom. Lett.* (2020).
- Yu, Chen and Kar, "Dynamic median consensus over random networks", in *IEEE 60th Conf. on Dec. and Control* (2021).

Literature

Definition: Dynamic consensus problem

Design the local interaction rules f_i such that the agents' state x_i converges to a scalar function $g : \mathbb{R}^n \to \mathbb{R}$ of the reference signals $u_1, \ldots, u_n$, i.e., there exists $\varepsilon \geq 0$ such that

$$\|x_i(k) - g(u_1(k), \ldots, u_n(k))\| \leq \varepsilon, \qquad k \geq k^*, \quad i \in \mathcal{V},$$

The **maximum** (*highest value of a data set*).

- Deplano, Franceschelli, Giua, "Discrete-time Dynamic consensus on the max value", in *15th European Workshop on Advanced Control and Diagnosis*, Springer (2021)
- Deplano, Franceschelli, and Giua, "Dynamic min and max consensus and size estimation of anonymous multi-agent networks", in *IEEE Trans. Autom. Control* (2021, in press).
- Sen, Sahoo, and Slingh, "Global max-tracking of multiple time-varying signals using a distributed protocol", in *IEEE Control and Sys. Lett.* (2022)

Main contributions

Contribution 1

A novel protocol to solve the dynamic max-consensus problem with the following features:

- **Robustness** to re-initialization;
- **Scalability** in large networks;
- **Self-tuning logic** for arbitrary small steady state error;
- **Boundedness** of the tracking error.

Contribution 2

A novel algorithm to track graph's parameters tracking problems in open time-varying networks:

- **Cardinality**, the number of agents in a network;
- **Radius/diameter**, the length of the minimum/maximum distance between the agents.

Applications

Possible applications of the proposed protocol range across different fields:

- Real-time monitoring in decentralized systems;

  Simpson-Porco and Bullo, "Distributed monitoring of voltage collapse sensitivity indices", in *IEEE Trans. Smart Grid* (2016)
- Network's parameter estimation in anonymous networks;

  Garin, Varagnolo, and Johansson, "Distributed estimation of diameter, radius and eccentricities in anonymous networks", in *3rd IFAC NecSys* (2012)

  Varagnolo, Pillonetto, and Schenato, "Distributed cardinality estimation in anonymous networks", in *IEEE Trans. Autom. Control* (2013)
- Online optimization in distributed systems;

  Jiang and Charalambous, "Distributed ADMM using finite-time exact ratio consensus in digraphs", in *European Control Conf.* (2021)

  Bastianello and Carli, "ADMM for Dynamic Average Consensus Over Imperfect Networks", in *9th IFAC Necsys* (2022)
- Distributed synchronization in wireless sensor networks;

  Z. Dengchang *et al.*, "Time synchronization in wireless sensor networks using max and average consensus protocol" (2013)
- Leader election in multi-agent systems;

  T. Borsche and S. A. Attia, "On leader election in multi-agent control systems" (2010)
- and many others...

Outline

The proposed protocol

STDMC Protocol: SELF-TUNING DYNAMIC MAX-CONSENSUS

(Input): Tuning parameters $\alpha^{\mathrm{MAX}} \geq \alpha^{\mathrm{MIN}} > 0$.

(Initialization): $x_i(0) \in \mathbb{R}$ for $i \in \mathcal{V}$;
$\alpha_i(0) \in \{\alpha^{\mathrm{MIN}}, \alpha^{\mathrm{MAX}}\}$ for $i \in \mathcal{V}$

(Execution): for $k = 1, 2, 3, \ldots$ each node i does

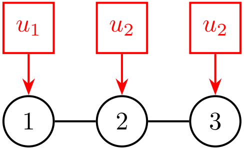
1) Gather $x_j(k-1)$ and $\alpha_j(k-1)$ from each neighbor $j \in \mathcal{N}_i$
2) Update the current state according to

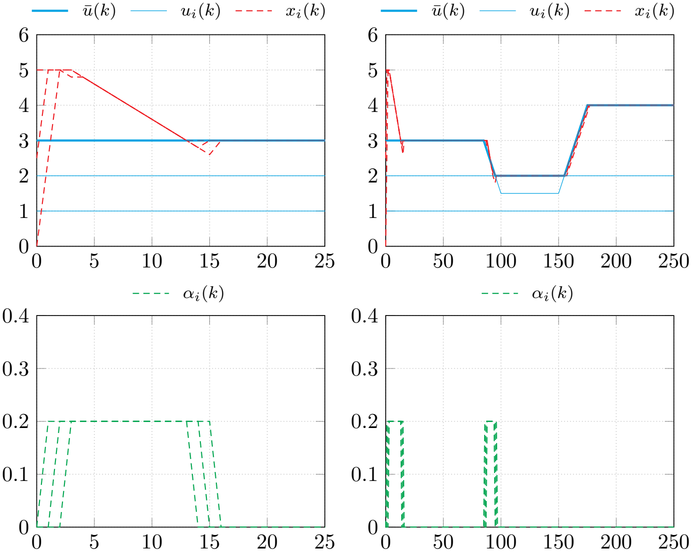
$$x_i(k) = \max_{j \in \mathcal{N}_i \cup \{i\}} \{x_j(k-1) - \alpha_j(k-1), u_i(k)\}$$

3) Update the current parameter according to

$$\alpha_i(k) = \begin{cases} \alpha^{\mathrm{MAX}} & \text{if } x_i(k) < x_i(k-1) \\ \alpha^{\mathrm{MIN}} & \text{otherwise} \end{cases}$$

A simple example





Initial conditions

$$u_1(0) = 1, \quad x_1(0) = 0, \quad \alpha_1(0) = 0$$
$$u_2(0) = 2, \quad x_2(0) = 2, \quad \alpha_2(0) = 0$$
$$u_3(0) = 3, \quad x_3(0) = 5, \quad \alpha_3(0) = 0$$

Parameters

$$\alpha^{\text{MAX}} = 0.2, \quad \alpha^{\text{MIN}} = 10^{-8}$$

Working assumption

Assumption 1

The variation of the reference signals $u_i(k)$ are bounded a constant $\Pi \geq 0$, i.e., for $k \geq 0$ it holds

$$\Delta u_i(k) = |u_i(k) - u_i(k-1)| \leq \Pi$$

Any continuous-time signal with bounded derivative can be over-sampled to reduce its absolute variation.

Main results

Theorem 1: Tracking error of STDMC Protocol

Consider time-varying reference signals $u_i(k) \in \mathbb{R}$ under Assumption 1 and let $\delta_{\mathcal{G}}$ be the diameter of graph $\mathcal{G}$. If $\mathcal{G}$ is connected and if

$$\alpha^{\text{MAX}} > \Pi, \tag{2}$$

then $\exists T_c \geq 0$ such that the tracking error $e_i(k)$ of each agent is bounded by ε_{tr}, i.e., for $k \geq T_c$ it holds

$$e_i(k) = |x_i(k) - \overline{u}(k)| \leq \varepsilon_{tr} = (\alpha^{\text{MAX}} + \Pi)\delta_{\mathcal{G}}, \qquad i \in \mathcal{V} \tag{3}$$

and moreover

$$T_c \leq \max\left\{\frac{\overline{x}(0) - \overline{u}(0)}{\alpha^{\text{MAX}} - \Pi}, \delta_{\mathcal{G}}\right\}. \tag{4}$$

Theorem 2: Steady-state error of STDMC Protocol

If the reference signals remain constant for $k \geq k_0$, then the steady state error $e_i(k)$ of each agent is bounded by ε_{ss}, i.e., for $k \geq k_0 + 2\delta_{\mathcal{G}}$ by ε_{ss},

$$e_i(k) = |x_i(k) - \overline{u}(k)| \leq \varepsilon_{ss} = \alpha^{\text{MIN}}\delta_{\mathcal{G}}, \qquad i \in \mathcal{V}. \tag{5}$$

Main results

Theorem 3: Tracking error when the bound Π is unknown

If the agents update their local parameter α_i^{MAX} according to

$$\alpha_i^{\text{MAX}}(k) = \max_{j \in \mathcal{N}_i \cup \{i\}} \{\alpha_j^{\text{MAX}}(k-1), \theta \cdot \Delta u_i(k)\}, \qquad \theta > 1, \tag{6}$$

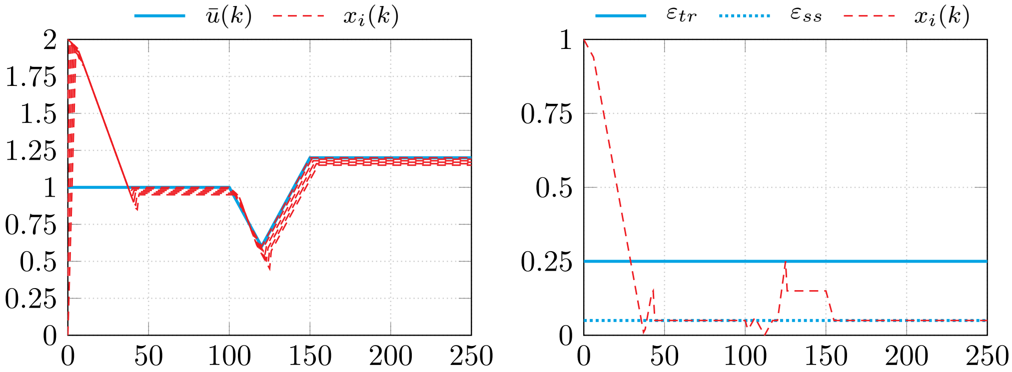
then $\exists T_c \geq 0$ such that the tracking error $e_i(k)$ is bounded for $k \geq T_c$ by the following

$$e_i(k) = |x_i(k) - \overline{u}(k)| \leq (\theta + 1)\Pi\delta_{\mathcal{G}} = \varepsilon_{tr}, \qquad i \in \mathcal{V}. \tag{7}$$

Outline

Piece-wise linear inputs - Line graph of 6 nodes

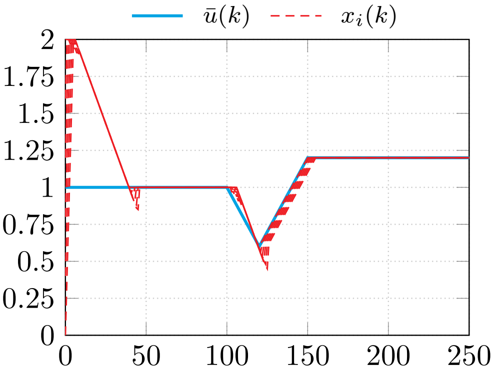
$\alpha^{\text{MAX}} = 0.03$ $\alpha^{\text{MIN}} = 0.01$ $\Pi = 0.02$ $\varepsilon_{tr} = 0.25$ $\varepsilon_{ss} = 0.05$

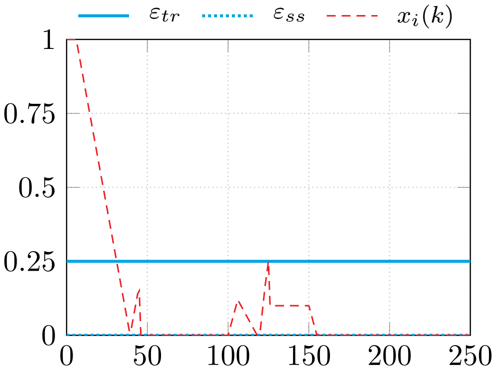


Piece-wise linear inputs - Line graph of 6 nodes

$\alpha^{\text{MAX}} = 0.03$ $\alpha^{\text{MIN}} = 1 \cdot 10^{-8}$ $\Pi = 0.02$ $\varepsilon_{tr} = 0.25$ $\varepsilon_{ss} = 5 \cdot 10^{-8}$

Arbitrary small steady state error!

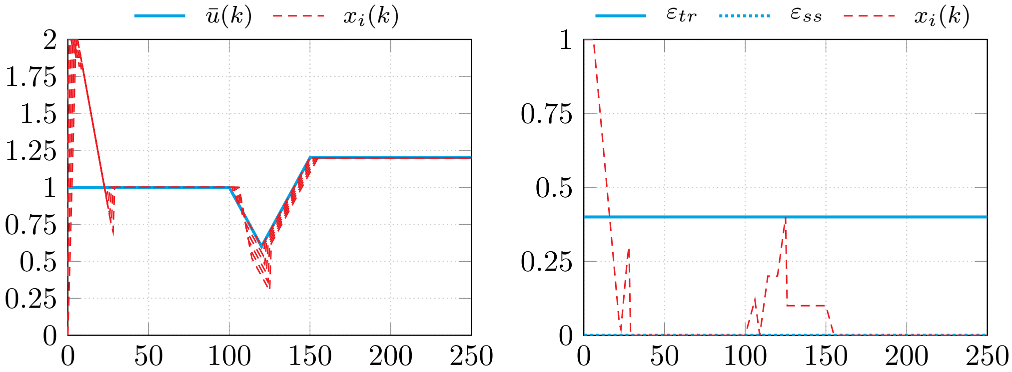




Piece-wise linear inputs - Line graph of 6 nodes

$\alpha^{\text{MAX}} = 0.06$ $\alpha^{\text{MIN}} = 1 \cdot 10^{-8}$ $\Pi = 0.02$ $\varepsilon_{tr} = 0.40$ $\varepsilon_{ss} = 5 \cdot 10^{-8}$

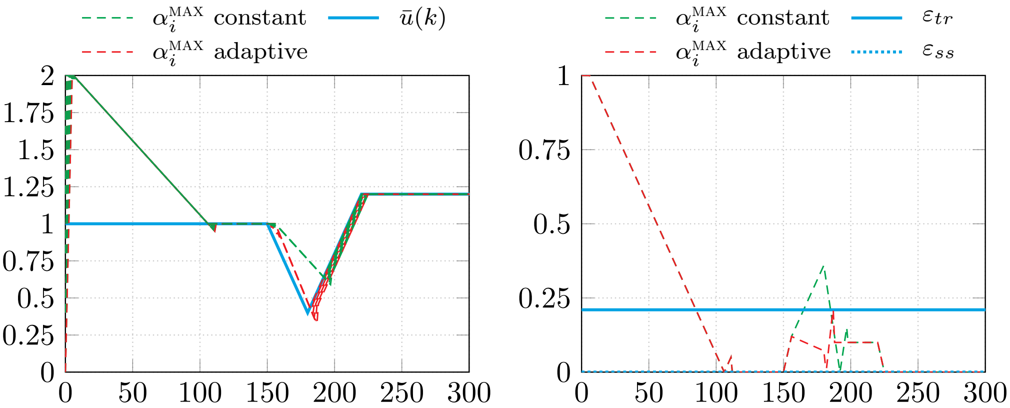
Faster convergence rate! But higher tracking error...



Unknown bound Π - Line graph of 6 nodes

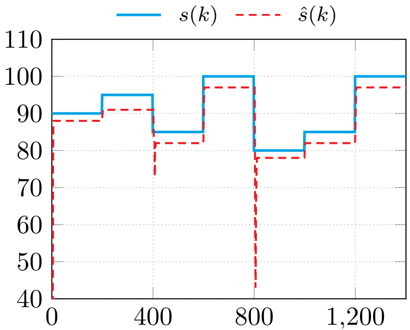
$\alpha_i^{\text{MAX}}(0) = 0.01$ $\alpha^{\text{MIN}} = 1 \cdot 10^{-8}$ $\Pi = 0.02$ $\theta = 1.1$ $\varepsilon_{tr} = 0.22$ $\varepsilon_{ss} = 5 \cdot 10^{-8}$

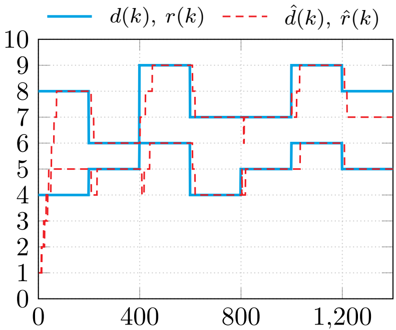
The assumption $\alpha_i^{\text{MAX}}(0) > \Pi$ is not satisfied!



Application to graph's parameters estimation - Random graph of $n \in [80, 100]$ nodes

$\alpha^{\mathrm{MAX}} = 0.1$, $\alpha^{\mathrm{MIN}} = 10^{-12}$, cardinality $s(k)$, diameter $d(k)$, radius $r(k)$





- Deplano, Franceschelli, and Giua, "Dynamic min and max consensus and size estimation of anonymous multi-agent networks", in *IEEE Trans. Autom. Control* (2021, in press).
- Deplano, Franceschelli, and Giua, "Distributed tracking of graph parameters in anonymous networks with time-varying topology", in *IEEE 60th Conf. on Dec. and Control* (2021).
- Varagnolo, Pillonetto, and Schenato, "Distributed cardinality estimation in anonymous networks", in *IEEE Trans. Autom. Control* (2013).

Outline

Conclusions and future directions

Contributions:

- The STDMC protocol solving the dynamic max-consensus (or max-tracking) problem;
- Theoretical characterization of tracking and steady-state errors, as well as convergence time;
- Employment to track time-varying graphs' parameters in open networks.

Potential extensions:

- Reference signals with unbounded variations;
- Noise in the measurements;
- Delays in the communications.

Future application perspectives:

- Core-periphery structure detection;
- Distributed online optimization algorithms;
- ...

Dynamic Max-Consensus with Local Self-Tuning

Thank you for your attention!

Diego Deplano

Email: diego.deplano@unica.it
Webpage: https://sites.google.com/view/deplanodiego/