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# Stability of Nonlinear Monotone Systems and Consensus in Multi-Agent Networks

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# Outline

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1. Introduction to Multi-Agent Systems (MASs)
2. Analysis of nonlinear MASs
3. Conclusions and future directions

# What is an agent?

*"Agents are computational systems that inhabit some complex dynamic environment, sense and act autonomously in this environment, and by doing so realize a set of goals or tasks for which they are designed."*

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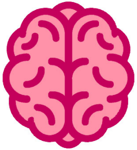
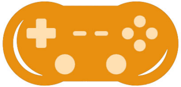
# What is an agent? Some examples

Self-driving car



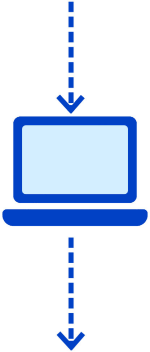
Human's opinion

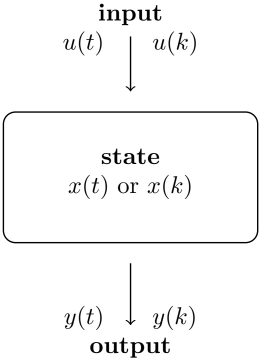


Player

Computational unit

# Model of the agent: dynamical system






An agent is modeled as a dynamical system with a state-space representation:

- Continuous time (CT): $\dot{x}(t) = f(y(t), u(t))$ and $y(t) = h(x(t))$ with $t \in \mathbb{R}$;
- Discrete time (DT): $x(k+1) = f(y(k), u(k))$ and $y(k) = h(x(k))$ with $k \in \mathbb{N}$.

# What is a multi-agent system?

*"A multi-agent system is a coupled network of agents that work together to find answers to problems that are beyond the individual capabilities or knowledge of each agent."*

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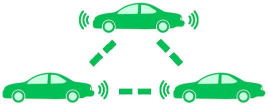
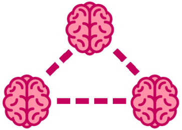
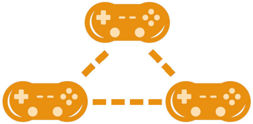


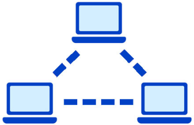
Multi-Robot control



Social opinion dynamics

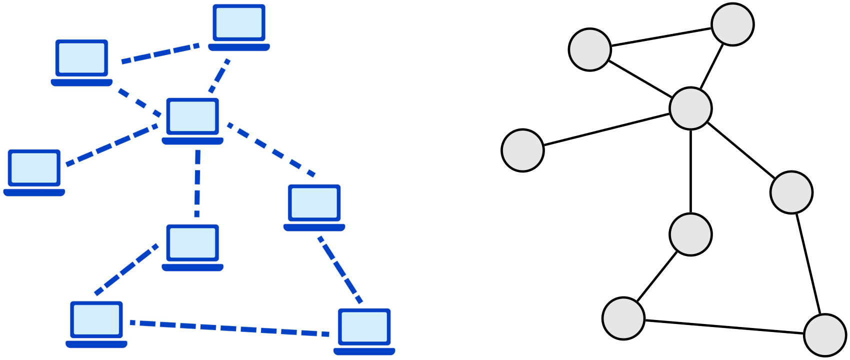


Game theory



Optimization theory

# Model of the network: graph



A network of agents is modeled with a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where:

- $\mathcal{V} \subset \mathbb{N}$ is the set of nodes (grey circles) modeling the agents;
- $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges (black lines) modeling the flow information among agents.

## Working assumptions

We consider Multi-Agents Systems (MASs) under the following working assumptions:

- Discrete time framework: $k \in \mathbb{N}$;
- A number of agents equal to $n \in \mathbb{N}$;
- Scalar agents: $x_i(k) \in \mathbb{R}$;
- Autonomous agents: $u_i(k) = 0$ for all $k$;
- Identity output map: $y_i(k) = x_i(k)$;
- Fixed directed interaction graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$;
- Nonlinear differentiable local interaction protocol $f_i : \mathbb{R}^n \to \mathbb{R}$;

Thus, the dynamics of each agent is

$$x_i(k+1) = f_i(x_i(k), x_j(k) : j \in \mathcal{N}_i), \quad \forall i \in \mathcal{V}, k \in \mathbb{N}. \tag{1}$$

while the dynamics of the overall system is

$$x(k+1) = f(x(k)), \quad \forall k \in \mathbb{N}. \tag{2}$$

## An example of multi-agent system modeling

Network → $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Set of agents → $\mathcal{V} = \{1, 2, 3\}$

Set of interactions → $\mathcal{E} \subseteq \{(1,2), (2,1), (2,3)\}$

Neighbors → $\mathcal{N}_1 = \{2\}, \; \mathcal{N}_2 = \{1,3\}, \; \mathcal{N}_3 = \varnothing$

State of agent $i$ → $x_i(k) \in \mathbb{R}$

State of the system → $x(k) \in \mathbb{R}^n$

Framework → Discrete time $k \in \mathbb{N}$


Linear interactions:

$$
\begin{aligned}
x_1(k+1) &= a_{1,1} \cdot x_1(k) + a_{1,2} \cdot x_2(k), \\
x_2(k+1) &= a_{2,1} \cdot x_1(k) + a_{2,2} \cdot x_2(k) + a_{1,2}a_{1,3} \cdot x_3(k), \\
x_3(k+1) &= a_{3,3} \cdot x_3(k).
\end{aligned}
$$

Let $A = \{a_{i,j}\}$ be the matrix formed by the coefficients $a_{i,j}$, then the MAS evolves according to

$$x(k+1) = f(x(k)) = Ax(k).$$

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Nonlinear interactions:

$$\begin{aligned} x_1(k+1) &= f_1(x_1(k), x_2(k)), \\ x_2(k+1) &= f_2(x_1(k), x_2(k), x_3(k)), \\ x_3(k+1) &= f_3(x_3(k)). \end{aligned}$$

Each function $f_i$, with $i \in \mathcal{V}$, is called the local interaction protocol of agent $i$, thus

$$x(k+1) = f(x(k)) = [f_1(\cdot), f_2(\cdot), f_3(\cdot)]^\top.$$

# Outline

## Main problem of interest

**Definition**: Consensus problem

The consensus problem consists in the design of a set of local interaction protocols $f_i$ such that each agent converges to the same constant state, i.e.,

$$\lim_{k \to \infty} x(k) \propto \mathbf{1},$$

for any initial condition $x(0) \in \mathbb{R}^n$.

Note that $\mathbf{1}^\top = [1, 1, \cdots, 1]^\top$ denotes a vector of ones of opportune dimension.

## Motivation and contribution

**Linear Multi-Agent Systems**

$$x(k+1) = Ax(k)$$

Classical Perron-Frobenius Theory deals with non-negative matrices $A$.

**Nonlinear Multi-Agent Systems**

$$x(k+1) = f(x(k))$$

Nonlinear Perron-Frobenius Theory deals with monotone and positive maps $f$.

For consensus more properties are needed

$A$ is row-stochastic

$+$

$A$ is indecomposable and aperiodic

$\Downarrow$

$$\lim_{k\to\infty} x(k) = \lim_{k\to\infty} A^k x(0) \propto \mathbf{1}$$

**?**

$$\lim_{k\to\infty} x(k) = \lim_{k\to\infty} f^k(x(0)) \propto \mathbf{1}$$

B. Lemmens and R. D. Nussbaum, "Nonlinear Perron-Frobenius Theory", 2012.

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A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be monotone if for all $x, y \in \mathbb{R}^n$ it holds

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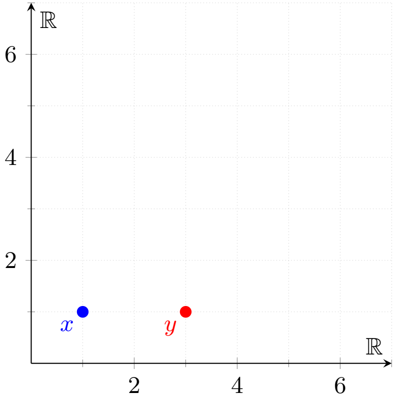

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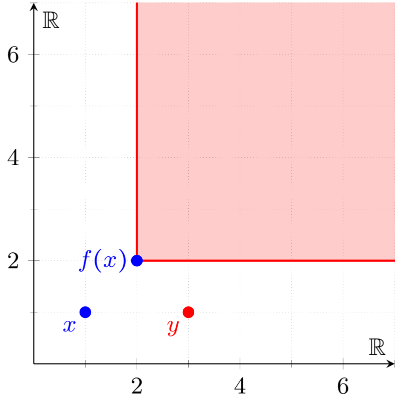

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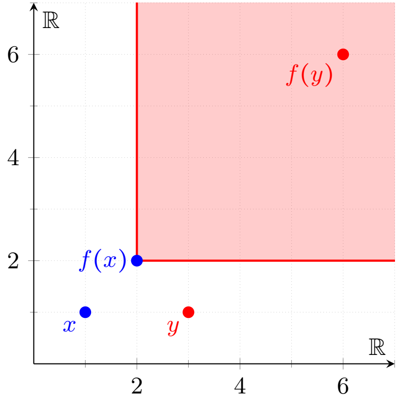

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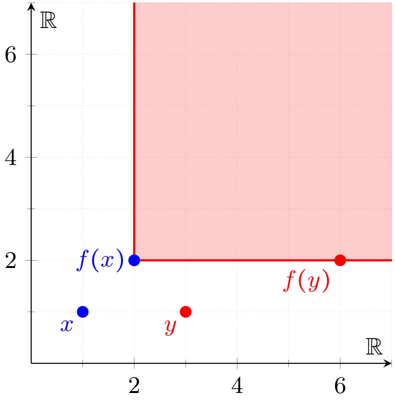

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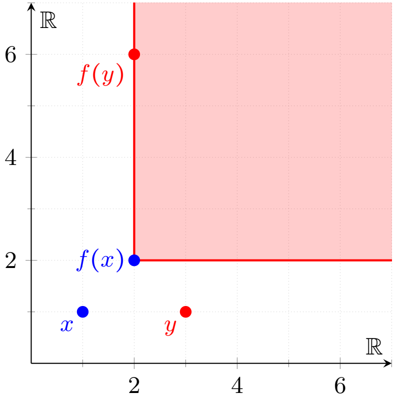

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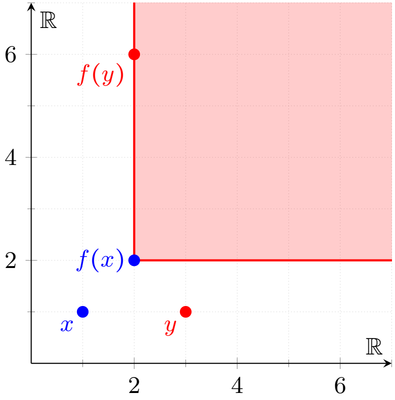
N.B. Monotonicity does not concern the trajectory! Consider the scalar system

$$x(k+1) = \alpha x(k), \quad \alpha \in (0,1),$$

then $f(x) < x$, $\forall x \geq 0$.



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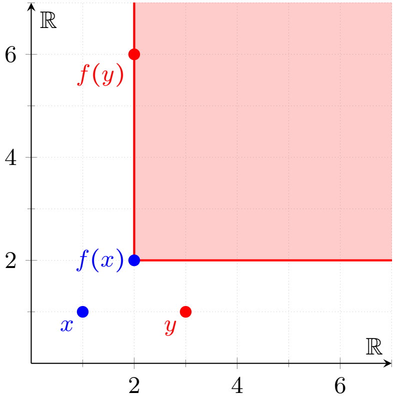


How to verify it?

## Solutions of monotone systems preserve the ordering between the initial conditions

**Example in $\mathbb{R}$**

Consider the scalar system

$$x(k+1) = x^3(k).$$

The solution for $x(0) = a$ is $f^k(a) = a^{2k}$. Thus, for two initial conditions $a \leq b$, it holds

$$f^k(a) = a^{3k} \leq b^{3k} = f^k(b), \qquad k \in \mathbb{N}.$$

Let $a = 1.1$ and $b = 1.2$, then the trajectories are

$$a) \quad \overbrace{1.1}^{k=0} \to \overbrace{1.33}^{k=1} \to \overbrace{2.36}^{k=2} \to \overbrace{13.11}^{k=3} \to \cdots$$

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**Definition**: positivity

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$$f(\mathbb{R}^n_{\geq 0}) \subseteq \mathbb{R}^n_{\geq 0}.$$

For a linear map $f(x) = Ax$, monotonicity is equivalent to positivity, in fact

$A$ is non-negative

$\swarrow\!\!\nearrow$ $\quad$ $\nwarrow\!\!\searrow$

$f(x)$ is monotone $\qquad$ $f(x)$ is positive.

A nonlinear monotone map $f(x)$ is positive if and only if $f(\mathbf{0}) \geq \mathbf{0}$.

Since we are looking for a stable consensus manifold, we will assume that

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**Linear Multi-Agent Systems**

$$x(k+1) = Ax(k)$$

**Nonlinear Multi-Agent Systems**

$$x(k+1) = f(x(k))$$

Conditions for reaching consensus

$A$ is non-negative
+
$A$ is row-stochastic
+
$A$ is indecomposable and aperiodic
$\Downarrow$
$$\lim_{k\to\infty} A^k x(0) \propto \mathbf{1}$$

$f$ is monotone and $f(\mathbf{0}) = \mathbf{0}$
+
$f$ is plus-homogeneous
+
?
$\Downarrow$
$$\lim_{k\to\infty} f^k(x(0)) \propto \mathbf{1}$$

# Outline

**Definition**: Plus-homogeneity

A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be plus-homogeneous if for all $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ it holds

$$f(x + \alpha\mathbf{1}) = f(x) + \alpha\mathbf{1}$$

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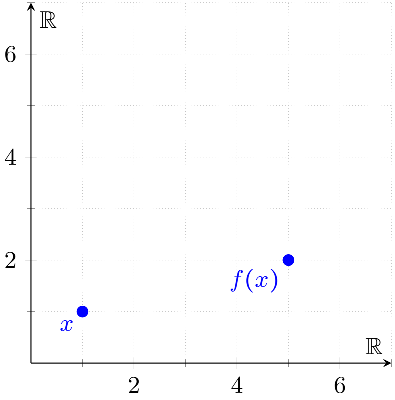

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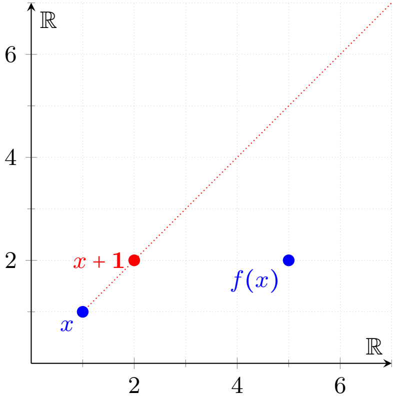

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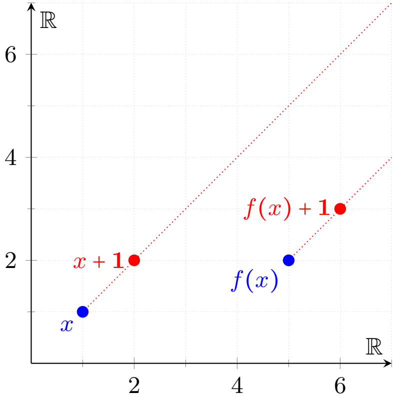

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**Definition**: Row-stochastic

Consider a linear map $f(x) = Ax : \mathbb{R}^n \to \mathbb{R}^n$. A non-negative matrix $A$ is said to be row-stochastic if all its row sums are equal to one, i.e.,

$$A\mathbf{1} = \mathbf{1}.$$

A linear map $f(x) = Ax$ is plus-homogeneous if and only if matrix $A$ is row-stochastic, in fact

$$f(x + \alpha \mathbf{1}) = Ax + \alpha A\mathbf{1} = Ax + \alpha \mathbf{1} = f(x) + \alpha \mathbf{1}.$$

**Definition**: Plus-homogeneity

A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be plus-homogeneous if for all $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ it holds

$$f(x + \alpha\mathbf{1}) = f(x) + \alpha\mathbf{1}$$

**Definition**: Row-stochastic

Consider a linear map $f(x) = Ax : \mathbb{R}^n \to \mathbb{R}^n$. A non-negative matrix $A$ is said to be row-stochastic if all its row sums are equal to one, i.e.,

$$A\mathbf{1} = \mathbf{1}.$$

A linear map $f(x) = Ax$ is plus-homogeneous if and only if matrix $A$ is row-stochastic, in fact

$$f(x + \alpha\mathbf{1}) = Ax + \alpha A\mathbf{1} = Ax + \alpha\mathbf{1} = f(x) + \alpha\mathbf{1}.$$

## Solutions of plus-homogeneous systems are invariant to rigid transformations

**Example in $\mathbb{R}^2$**

Consider the system $[x,\ y]^\top \in \mathbb{R}^2$

$$x(k+1) = \max\{x(k)+1, y(k)-2\}, \qquad y(k+1) = x(k)+2.$$

This function is plus-homogeneous since for any initial condition $a = [a_1, a_2]^\top \in \mathbb{R}^2$ satisfies

$$f(a+\alpha\mathbf{1}) = \begin{bmatrix} \max\{a_1+1, a_2-2\}+\alpha \\ a_1+2+\alpha \end{bmatrix} = f(a)+\alpha, \qquad \forall \alpha \in \mathbb{R}, k \in \mathbb{N}.$$

Let $a = [1,5]^\top$ and $b = a+\mathbf{1} = [2,6]^\top$, then the trajectories are

$$a) \quad \overbrace{\begin{bmatrix}1\\5\end{bmatrix}}^{k=0} \to \overbrace{\begin{bmatrix}3\\3\end{bmatrix}}^{k=1} \to \overbrace{\begin{bmatrix}4\\5\end{bmatrix}}^{k=2} \to \overbrace{\begin{bmatrix}5\\6\end{bmatrix}}^{k=3} \to \cdots$$

$$b) \quad \underbrace{\begin{bmatrix}2\\6\end{bmatrix}}_{k=0} \to \underbrace{\begin{bmatrix}4\\4\end{bmatrix}}_{k=1} \to \underbrace{\begin{bmatrix}5\\6\end{bmatrix}}_{k=2} \to \underbrace{\begin{bmatrix}6\\7\end{bmatrix}}_{k=3} \to \cdots$$

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| **Linear Multi-Agent Systems** | **Nonlinear Multi-Agent Systems** |
|---|---|
| $x(k+1) = Ax(k)$ | $x(k+1) = f(x(k))$ |

<u>Conditions for reaching consensus</u>

| | |
|---|---|
| $A$ is non-negative | $f$ is monotone and $f(\mathbf{0}) = \mathbf{0}$ |
| $+$ | $+$ |
| $A$ is row-stochastic | $f$ is plus-homogeneous |
| $+$ | $+$ |
| $A$ is indecomposable and aperiodic | **?** |
| $\Downarrow$ | $\Downarrow$ |
| $\lim_{k\to\infty} A^k x(0) \propto \mathbf{1}$ | $\lim_{k\to\infty} f^k(x(0)) \propto \mathbf{1}$ |

# Outline

**Definition**: Indecomposability and aperiodicity

A row-stochastic matrix $A \in \mathbb{R}^n \times \mathbb{R}^n$ is said to be indecomposable and aperiodic if

$$A_\infty = \lim_{k \to \infty} A^k$$

exists and all the rows of $A_\infty$ are the same.

**Definition**: Graph associated to a linear map

Consider a linear map $f(x) = Ax : \mathbb{R}^n \to \mathbb{R}^n$. The graph $\mathcal{G}(A) = (\mathcal{V}, \mathcal{E})$ associated to matrix $A = \{a_{i,j}\}$ is defined by:

- A set of nodes $\mathcal{V} = \{1, \dots, n\}$;
- A set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, such that

$$(i,j) \in \mathcal{E} \quad \text{if} \quad a_{ij} \neq 0,$$

where $a_{ij}$ denotes the element in the $i$-th row and $j$-th column of matrix A.

J. Wolfowitz, "Products of indecomposable, aperiodic, stochastic matrices", 1963.

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**Definition**: Connectivity properties of directed graphs

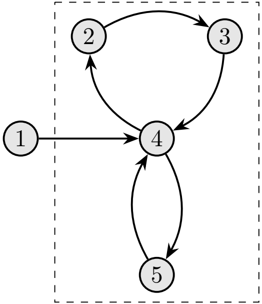
- A directed graph $\mathcal{G}$ is strongly connected if there exists a directed path from any node to any other node;
- A directed graph $\mathcal{G}$ has a sink component if there exists a subgraph which is strongly connected and have not any outgoing edge.
- A sink component is said to be:
  - globally reachable if it can be reached from any other node by traversing a directed path;
  - aperiodic if the greatest common divisor of the lengths of all its cycles is equal to one.

$A$ is indecomposabile and aperiodic

$$\Updownarrow$$

$\mathcal{G}(A)$ has a globally reachable and aperiodic sink component.

Question: How a graph is defined in terms of a nonlinear map $f$?



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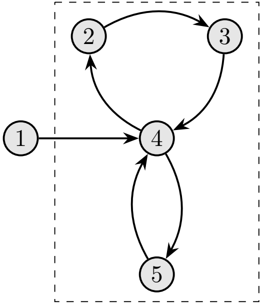


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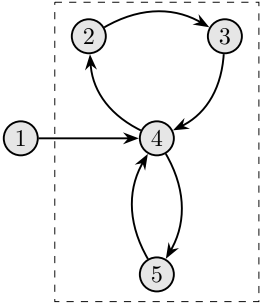


Question: How a graph is defined in terms of a nonlinear map $f$?

**Definition**: Graph associated to a nonlinear map which is differentiable almost everywhere

Given a nonlinear map $f : \mathbb{R}^n \to \mathbb{R}^n$, let $J(x) = \{J_{i,j}(x)\}$ be its jacobian matrix at $x \in \mathbb{R}^n$, i.e.,

$$J_{i,j}(x) = \frac{\partial f_i}{\partial x_j}$$

Then, the associated graph $\mathcal{G}(f) = (\mathcal{V}, \mathcal{E})$ is defined by:

- A set of nodes $\mathcal{V} = \{1, \dots, n\}$;
- A set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, such that

$$(i, j) \in \mathcal{E} \quad \text{if} \quad J_{i,j}(x) \neq 0, \quad \forall x \in \mathbb{R}^n \setminus S,$$

where $S$ is a set of measure zero.

| **Linear Multi-Agent Systems** | **Nonlinear Multi-Agent Systems** |
| --- | --- |
| $x(k+1) = Ax(k)$ | $x(k+1) = f(x(k))$ |

Conditions for reaching consensus

| | |
| --- | --- |
| $A$ is non-negative | $f$ is monotone and $f(\mathbf{0}) = \mathbf{0}$ |
| $+$ | $+$ |
| $A$ is row-stochastic | $f$ is plus-homogeneous |
| $+$ | $+$ |
| $\mathcal{G}(A)$ has a sink component being globally reachable and aperiodic | $\mathcal{G}(f)$ has a sink component being globally reachable and aperiodic |
| $\Downarrow$ | $\Downarrow$ |
| $\lim_{k\to\infty} A^k x(0) \propto \mathbf{1}$ | $\lim_{k\to\infty} f^k(x(0)) \propto \mathbf{1}$ |

This remains as a conjecture!

**Linear Multi-Agent Systems**

$$x(k+1) = Ax(k)$$

**Nonlinear Multi-Agent Systems**

$$x(k+1) = f(x(k))$$

Conditions for reaching consensus

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+
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⇓
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$f$ is type-K monotone and $f(\mathbf{0}) = \mathbf{0}$
+
$f$ is plus-homogeneous
+
$\mathcal{G}(f)$ has a sink component being globally reachable ~~and aperiodic~~
⇓
$$\lim_{k\to\infty} f^k(x(0)) \propto \mathbf{1}$$

This has been proved!

# Outline

**Definition**: Type-K monotonicity

An monotone map $f:\mathbb{R}^n \to \mathbb{R}^n$ is said to be type-K if for all $x \leq y$ such that $x \neq y$ it holds:

$$x_i < y_i \Rightarrow f_i(x) < f_i(y), \qquad \forall i \in \{1,\dots,n\}$$

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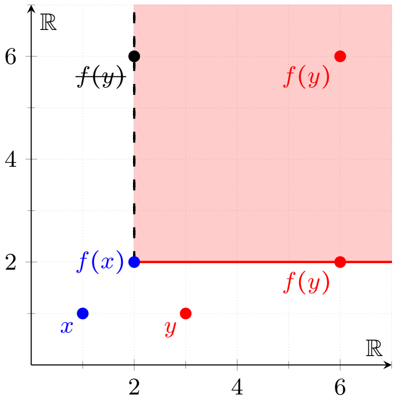

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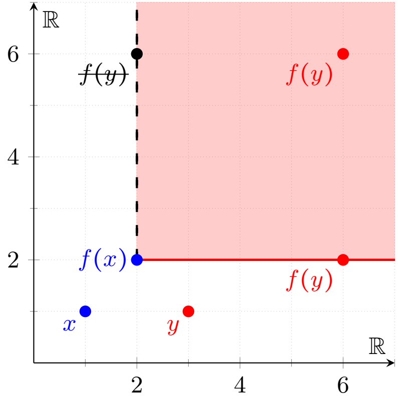

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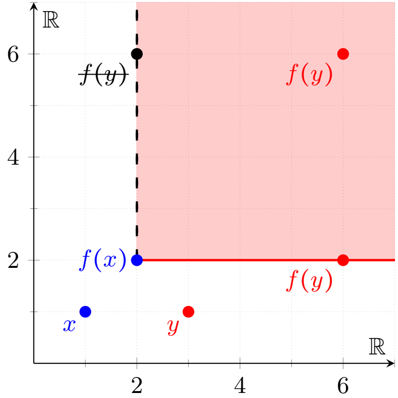


How to verify it?

**Definition**: Strong monotonicity

An monotone map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be strong if for all $x \leq y, x \neq y$ it holds:

$$f_i(x) < f_i(y), \qquad \forall i \in \{1, \dots, n\}$$

N.B. Type-K monotonicity is more general than strong monotonicity, for which a vast literature already exists.

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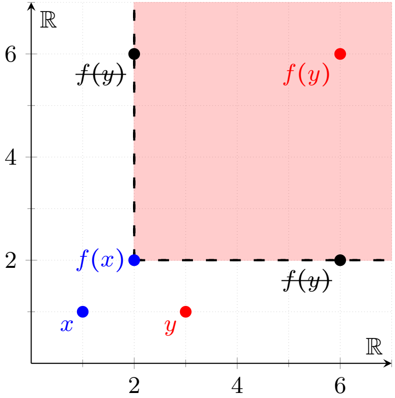

STRONG MONOTONICITY $\Rightarrow$ TYPE-K MONOTONICITY $\Rightarrow$ MONOTONICITY

**Hint**: Every converse relations does not hold

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, then:

- $f(x, y) = (y, x)$ is monotone but not type-K monotone;
- $g(x, y) = ((x + y)/2, y)$ is type-K monotone but not strongly monotone.

**Hint**: Type-K monotonicity rules out periodic behavior

Consider the systems

$$x(k+1) = f(x(k)), \qquad y(k+1) = g(y(k)),$$

Let the same initial condition $x(0) = y(0) = [1,\ 2]^\top$, the trajectories of the systems are

$$x(k): \quad x(0) = \begin{bmatrix}1\\2\end{bmatrix} \to \begin{bmatrix}2\\1\end{bmatrix} \to \begin{bmatrix}1\\2\end{bmatrix} \to \begin{bmatrix}2\\1\end{bmatrix} \to \cdots (\textit{periodic point of period } p = 2)$$

$$y(k): \quad y(0) = \begin{bmatrix}1\\2\end{bmatrix} \to \begin{bmatrix}1.5\\2\end{bmatrix} \to \begin{bmatrix}1.75\\2\end{bmatrix} \to \begin{bmatrix}1.875\\2\end{bmatrix} \to \cdots \to \begin{bmatrix}2\\2\end{bmatrix} \quad (\textit{fixed point})$$

Note that in these examples the maps $f$ and $g$ are also plus-homogeneous.

**Linear Multi-Agent Systems**

$$x(k+1) = Ax(k)$$

**Nonlinear Multi-Agent Systems**

$$x(k+1) = f(x(k))$$

Conditions for reaching consensus

Linear:

$A$ is non-negative

$+$

$A$ is row-stochastic

$+$

$\mathcal{G}(A)$ has a sink component being globally reachable and aperiodic

$\Downarrow$

$$\lim_{k\to\infty} A^k x(0) \propto \mathbf{1}$$

Nonlinear:

$f$ is type-K monotone and $f(\mathbf{0}) = \mathbf{0}$

$+$

$f$ is plus-homogeneous

$+$

$\mathcal{G}(f)$ has a sink component being globally reachable

$\Downarrow$

$$\lim_{k\to\infty} f^k(x(0)) \propto \mathbf{1}$$

# Outline

**Theorem**: Convergence to consensus

Let map $f : \mathbb{R}^n \to \mathbb{R}^n$ be type-K monotone and plus-homogeneous and let $f(\mathbf{0}) = \mathbf{0}$. If graph $\mathcal{G}(f)$ possesses a globally reachable node, then all trajectories converge to a consensus, i.e.,

$$\lim_{k\to\infty} f^k(x) \propto \mathbf{1} \quad \forall x \in \mathbb{R}^n.$$

**Proof Sketch:**

1. If the map $f$ is monotone and plus-homogeneous then it nonexpansive w.r.t. the sup-norm, namely
$$\|f(x) - f(y)\|_\infty \leq \|x - y\|_\infty, \qquad \forall x, y \in \mathbb{R}^n.$$
2. Trajectories generated by sup-norm nonexpansive maps either are all unbounded, or all converge to some periodic orbit.
3. Type-K monotonicity prevents periodic orbit.
4. Since there is at least a fixed point $f(\mathbf{0}) = \mathbf{0}$, then the consensus points are fixed pointsx.
5. If the graph possesses a globally reachable node, then the only fixed points of $f$ are the consensus points.

Problem: How to apply this result in Multi-Agent Systems?

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**Theorem**: Local criteria for consensus in discrete-time

Consider a discrete-time MAS, where agents have dynamics

$$x_i(k+1) = f_i(x_i(k), x_j(k) : j \in \mathcal{N}_i).$$

If the set of differentiable local interaction rules $f_i : \mathbb{R}^n \to \mathbb{R}$, with $i = 1, \dots, n$, satisfy the next conditions:

1. $\partial f_i / \partial x_i > 0$ and $\partial f_i / \partial x_j \geq 0$ for $i \neq j$ (*type-K monotonicity*);
2. $f_i(x + \alpha \mathbf{1}) = f_i(x) + \alpha$ for any $\alpha \in \mathbb{R}$ (*plus-homogeneity*);
3. $f_i(\mathbf{0}) = 0$ (*positivity*);
4. Graph $\mathcal{G}(f)$ possesses a globally reachable node;

then the MAS converges asymptotically to a consensus state for any initial state $x(0) \in \mathbb{R}$.

D. Deplano, M. Franceschelli, and A. Giua, "A nonlinear Perron–Frobenius approach for stability and consensus of discrete-time multi-agent systems", in Automatica (2021).

D. Deplano, M. Franceschelli, and A. Giua, "Novel Stability Conditions for Nonlinear Monotone Systems and Consensus in Multi-Agent Networks", in IEEE Transaction on Automatic Control (2023).

**Discrete-time Systems**

Kamke-like condition

The map $f$ of a discrete-time system

$$x(k+1) = f(x(k))$$

is type-K monotone if and only if its Jacobian matrix is Metzler with strictly positive diagonal elements,

$$\frac{\partial f_i}{\partial x_i} > 0, \quad \frac{\partial f_i}{\partial x_j} \geq 0 \quad for \quad i \neq j$$

**Continuous-time Systems**

Kamke condition

The map $f$ of a continuous-time system

$$\dot{x}(t) = f(x(t))$$

is monotone if and only if its Jacobian matrix is Metzler,

$$\frac{\partial f_i}{\partial x_j} \geq 0 \quad for \quad i \neq j$$

E. Kamke. "*Zur Theorie der Systeme gewöhnlicher Differentialgleichungen. II.*" Acta Mathematica, 1932."

# Outline

**Theorem**

Consider a continuous-time dynamical system $\dot{x}(t) = f(x(t))$ such that $f$ is $C^1$. Then, the system is monotone if and only if the system is type-K monotone.

If $f$ is not $C^1$, then monotonicity $\not\Rightarrow$ type-K monotone:

$$\dot{x}(t) = -\text{sign}(x), \quad \text{with solution} \quad \varphi(t, x_0) = \begin{cases} x_0 - \text{sign}(x_0) \cdot t & \text{if } t < |x_0| \\ 0 & \text{if } t \geq |x_0| \end{cases}, \quad \text{for} \quad x(0) = x_0.$$

If the system evolves in discrete-time, then monotonicity $\not\Rightarrow$ type-K monotone:

$$x(k+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} x(k), \quad \text{with solution} \quad \varphi(k, x_0) = \begin{cases} x_0 & \text{if } k \text{ is odd} \\ Ax_0 & \text{if } k \text{ is even} \end{cases}, \quad \text{for} \quad x(0) = x_0.$$

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Consider a continuous-time dynamical system $\dot{x}(t) = f(x(t))$ such that $f$ is $C^1$. Then, the system is monotone if and only if the system is type-K monotone.

If $f$ is not $C^1$, then monotonicity $\not\Rightarrow$ type-K monotone:

$$\dot{x}(t) = -\text{sign}(x), \quad \text{with solution} \quad \varphi(t, x_0) = \begin{cases} x_0 - \text{sign}(x_0) \cdot t & \text{if } t < |x_0| \\ 0 & \text{if } t \geq |x_0| \end{cases}, \quad \text{for} \quad x(0) = x_0.$$

If the system evolves in discrete-time, then monotonicity $\not\Rightarrow$ type-K monotone:

$$x(k+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} x(k), \quad \text{with solution} \quad \varphi(k, x_0) = \begin{cases} x_0 & \text{if } k \text{ is odd} \\ Ax_0 & \text{if } k \text{ is even} \end{cases}, \quad \text{for} \quad x(0) = x_0.$$

**Discrete-time Systems**

Kamke-like condition

The map $f$ of a discrete-time system

$$x(k+1) = f(x(k))$$

is type-K monotone if and only if
its Jacobian matrix is Metzler with strictly positive
diagonal elements,

$$\frac{\partial f_i}{\partial x_i} > 0, \quad \frac{\partial f_i}{\partial x_j} \geq 0 \quad for \quad i \neq j$$

**Continuous-time Systems**

Kamke condition

The map $f$ of a continuous-time system

$$\dot{x}(t) = f(x(t))$$

is type-K monotone if and only if
its Jacobian matrix is Metzler,

$$\frac{\partial f_i}{\partial x_j} \geq 0 \quad for \quad i \neq j$$

E. Kamke. "*Zur Theorie der Systeme gewöhnlicher Differentialgleichungen. II.*" Acta Mathematica, 1932."

**Theorem**: Local criteria for consensus in continuous-time

Consider a continuous-time MAS, where agents have dynamics

$$\dot{x}_i(t) = f_i(x_i(t), x_j(t) : j \in \mathcal{N}_i).$$

If the set of differentiable local interaction rules $f_i : \mathbb{R}^n \to \mathbb{R}$, with $i = 1, \dots, n$, satisfy the next conditions:

1. $\partial f_i / \partial x_j \geq 0$ for $i \neq j$ (*type-K monotonicity*);
2. $f_i(x + \alpha \mathbf{1}) = f_i(x)$ for any $\alpha \in \mathbb{R}$ (*plus-homogeneity*);
3. $f_i(x) = 0$ if $x_i = x_j$ for all $j \in \mathcal{N}_i$ (*positivity*);
4. Graph $\mathcal{G}(f)$ possesses a globally reachable node;

then the MAS converges asymptotically to a consensus state for any initial state $x(0) \in \mathbb{R}$.

D. Deplano, M. Franceschelli, and A. Giua, "A nonlinear Perron–Frobenius approach for stability and consensus of discrete-time multi-agent systems", in Automatica (2021).

D. Deplano, M. Franceschelli, and A. Giua, "Novel Stability Conditions for Nonlinear Monotone Systems and Consensus in

Multi-Agent Networks", in IEEE Transaction on Automatic Control (2023).

# Outline

# A general class of nonlinear protocols for achieving consensus

The proposed theory apply to all networks of single integrator agents with dynamics

$$x_i(k+1) = x_i(k) + \varepsilon_i \sum_{j \in \mathcal{N}_i} h_i\left(x_j(k) - x_i(k)\right)$$

where the functions $h_i : \mathbb{R} \to \mathbb{R}$ satisfy the following

1. $h_i(\mathbf{0}) = 0$
2. $\frac{d}{dx} h_i(x) \geq 0$ with $\varepsilon_i < \left(|\mathcal{N}_i| \frac{d}{dx} h_i(x)\right)^{-1}$ for all $x \in \mathbb{R}$;

## Remarks

1. Functions $h_i$ can be heterogeneous among the agents;
2. Note that plus-homogeneity always holds since the functions take as input only state-differences of the agents; in fact, $(x_j(k) + \alpha) - (x_i(k) + \alpha) = x_j(k) - x_i(k)$;
3. If the functions $h_i$ are taken as the identity map $h_i(x) = x$ and $\varepsilon = \varepsilon_i$, then the protocol reduces to the well-known Laplacian dynamics $x(k+1) = (I - \varepsilon L)x(k)$.

# Example: Bounded control input

Consider the following dynamics

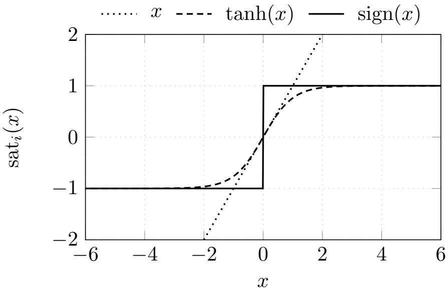
$$x_i(k+1) = x_i(k) + \varepsilon_i \sum_{j \in \mathcal{N}_i} h_i\left(x_j(k) - x_i(k)\right)$$

where $h_i$ are saturating function of the form

$$\mathsf{sat}_i(x) = \left(\frac{1 - e^{-m_i x}}{1 + e^{-m_i x}}\right), \qquad m_i \geq 0.$$

Such saturating functions encompass several well-known saturating functions, notably:

- $h_i = \mathsf{tanh}$ if $m_i = 2$;
- $h_i \approx \mathsf{sign}$ if $m_i \to \infty$.



# Example: Bounded control input

**Theorem**: Consensus with bounded input

A network of $n$ agents with dynamics

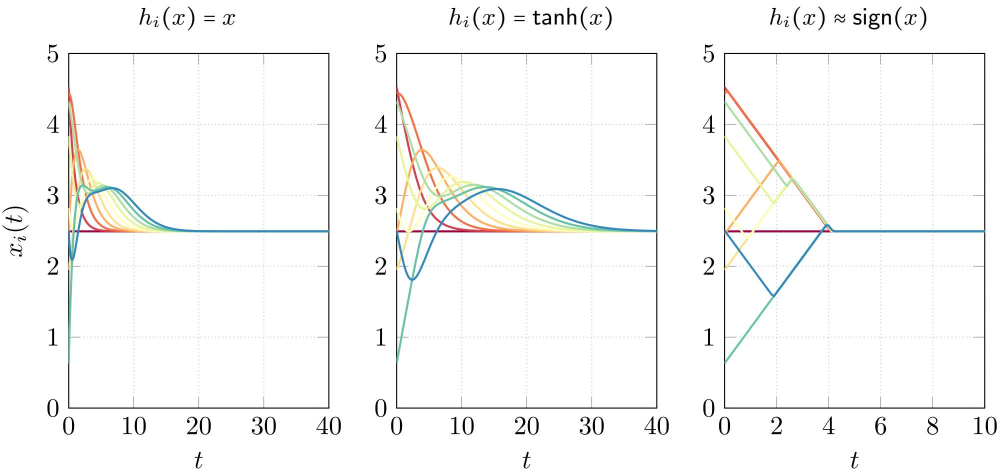
$$x_i(k+1) = x_i(k) + \varepsilon_i \sum_{j \in \mathcal{N}_i} \left( \frac{1 - e^{-m_i(x_j(k) - x_i(k))}}{1 + e^{-m_i(x_j(k) - x_i(k))}} \right), \qquad s_i, m_i \geq 0$$

achieve consensus asymptotically for all $x(0) \in \mathbb{R}^n$ if:

- Parameter $\varepsilon_i$ satisfies $\varepsilon_i < \frac{2}{m_i |\mathcal{N}_i|}$;
- Graph $\mathcal{G}(f)$ possesses a globally reachable node.

- Type-K monotonicity is satisfied since
  - $\partial f_i / \partial x_j \geq 0$ for all $j \neq i$;
  - $\partial f_i / \partial x_i > 0$ if it holds $\varepsilon_i < \frac{2}{m_i |\mathcal{N}_i|}$;
- Plus-homogeneity is satisfied since the local control action takes into account only state differences, in fact $(x_j(k) + \alpha) - (x_i(k) + \alpha) = x_j(k) - x_i(k)$;
- Positivity is satisfied since $h_i(0) = 0$.

# Example: Bounded control input - 10 agents in a line network



# Continuous-time: chemical reaction networks

Consider the following set of chemical reactions [R0] representing an enzymatic futile cycle

$$\dot{s}(t) = \Gamma h(s(t)), \qquad \text{with} \qquad \Gamma = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad h(s) = \begin{bmatrix} k_2 s_3 s_1 - k_1 s_5 \\ k_3 s_5 \\ k_4 s_4 s_2 - k_5 s_6 \\ k_6 s_6 \end{bmatrix}, \quad k_i \geq 0.$$

Noting that $s(t) \geq 0$, we study the system in the space of "*reaction coordinates*" $x(t)$

$$\dot{x}(t) = h(\sigma + \Gamma x(t)), \qquad \text{with} \qquad s(t) = \sigma + \Gamma x(t), \quad \text{and} \quad \sigma \in \mathbb{R}^n_{\geq 0},$$

that is such that:

- $\Gamma$ has zero row-sums, i.e., $\Gamma \mathbf{1} = \mathbf{0}$ (plus-homogeneity);
- The Jacobian of $h(\sigma + \Gamma x(t))$ is Metzler (type-K monotonicity), indeed:

$$J_h = \left.\frac{\partial h}{\partial x}\right|_{\sigma + \Gamma x = s} = \begin{bmatrix} * & k_1 + k_2 s_1 & 0 & k_2 s_3 \\ k_3 & * & 0 & 0 \\ 0 & k_4 s_4 & * & k_5 + k_4 s_2 \\ 0 & 0 & k_6 & * \end{bmatrix}.$$

[R0] Angeli and Sontag, "*Translation-invariant monotone systems, and a global convergence result for enzymatic futile cycles*", in Nonlinear Analysis: Real World App., (2008).

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# Outline

# Main contributions

## Contribution for general dynamical systems

Introduction of *type-K monotonicity* and its characterization for smooth systems:

- Type-K monotonicity can be verified by the sign-structure of the Jacobian matrix (see Theorem 5 in [R1] see Proposition 9 in [R2]);
- Trajectories of smooth type-K monotone and plus-homogeneous systems asymptotically converge toward equilibrium points (see Theorem 1 in [R1] and Theorem 13 in [R2]);
- Smooth monotone systems in continuous-time are also type-K monotone (see Theorem 3 in [R1]), while this is not true in discrete-time (see Remark 7 in [R2] and Proposition 1 in [R1]).

## Contribution for Multi-Agent Systems (MASs)

Application to consensus:

- A MAS achieves consensus asymptotically if it is type-K monotone, if it is plus-homogeneous, and if the graph contains a globally reachable node (see Theorem 6 in [R1] and Theorem 14 in [R2] for the discrete-time case and Corollary 1 in [R1] for the continuous-time case).

[R1] D. Deplano, M. Franceschelli, and A. Giua, "Novel Stability Conditions for Nonlinear Monotone Systems and Consensus in Multi-Agent Networks", in IEEE Transaction on Automatic Control (2023).
[R2] D. Deplano, M. Franceschelli, and A. Giua, "A nonlinear Perron–Frobenius approach for stability and consensus of discrete-time multi-agent systems", in Automatica (2021).

# Future directions

The theory developed in the first part paves fertile lines of research:

- Consider non differentiable maps;
- Consider time-varying maps;
- Consider nonlinear spaces;
- Consider non autonomous agents;
- Consider open networks
- And many more...

Future application perspectives:

- Distributed online optimization algorithms;
- Coordination schemes in multi-robot systems;
- Monotone games in game theory;
- ...

# Stability of Nonlinear Monotone Systems and Consensus in Multi-Agent Networks

Thank you for your attention!

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