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Coordination of multi-agent systems: stability via nonlinear Perron-Frobenius theory and consensus for desynchronization and dynamic estimation

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Declaration of Authorship

I, Diego Deplano, declare that this thesis titled '*Coordination of multi-agent systems: stability via nonlinear Perron-Frobenius theory and consensus for desynchronization and dynamic estimation*' and the work presented in it are my own. I confirm that:

- This work was done wholly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

I, Diego Deplano, gratefully acknowledge Sardinian Regional Government for the financial support of my PhD scholarship (P.O.R. Sardegna F.S.E. - Operational Programme of the Autonomous Region of Sardinia, European Social Fund 2014-2020 - Axis III Education and training, Thematic goal 10, Investment Priority 10ii), Specific goal 10.5.

Signed: Diego Deplano

Date: 18/02/2021

Dedicated to my family.

Abstract

This thesis addresses a variety of problems that arise in the study of complex networks composed by multiple interacting agents, usually called multi-agent systems (MASs). Each agent is modeled as a dynamical system whose dynamics is fully described by a state-space representation.

In the first part the focus is on the application to MASs of recent results that deal with the extensions of Perron-Frobenius theory to nonlinear maps. In the shift from the linear to the nonlinear framework, Perron-Frobenius theory considers maps being order-preserving instead of matrices being nonnegative. The main contribution is threefold. First of all, a convergence analysis of the iterative behavior of two novel classes of order-preserving nonlinear maps is carried out, thus establishing sufficient conditions which guarantee convergence toward a fixed point of the map: nonnegative row-stochastic matrices turns out to be a special case. Secondly, these results are applied to MASs, both in discrete and continuous-time: local properties of the agents' dynamics have been identified so that the global interconnected system falls into one of the above mentioned classes, thus guaranteeing its global stability. Lastly, a sufficient condition on the connectivity of the communication network is provided to restrict the set of equilibrium points of the system to the consensus points, thus ensuring the agents to achieve consensus. These results do not rely on standard tools (e.g., Lyapunov theory) and thus they constitute a novel approach to the analysis and control of multi-agent dynamical systems.

In the second part the focus is on the design of dynamic estimation algorithms in large networks which enable to solve specific problems. The first problem consists in breaking synchronization in networks of diffusively coupled harmonic oscillators. The design of a local state feedback that achieves desynchronization in connected networks with arbitrary undirected interactions is provided. The proposed control law is obtained via a novel protocol for the distributed estimation of the Fiedler vector of the Laplacian matrix. The second problem consists in the estimation of the number of active agents in networks wherein agents are allowed to join or leave. The adopted strategy consists in the distributed and dynamic estimation of the maximum among numbers locally generated by the active agents and the subsequent inference of the number of the agents that took part in the experiment. Two protocols are proposed and characterized to solve the consensus problem on the time-varying max value. The third problem consists in the average state estimation of a large network of agents where only a few agents' states are accessible to a centralized observer. The proposed strategy projects the dynamics of the original system into a lower dimensional state space, which is useful when dealing with large-scale systems. Necessary and sufficient conditions for the existence of a linear and a sliding mode observers are derived, along with a characterization of their design and convergence properties.

Sommario

In questa tesi si affrontano vari problemi che sorgono nello studio di reti complesse in cui una molteplicità di agenti interagisce tra loro, di solito denominati sistemi multi-agente. Ogni agente è modellato come un sistema dinamico con rappresentazione in variabili di stato.

Nella prima parte, l'attenzione è posta sull'applicazione ai sistemi multi-agente di nuove recenti estensioni della teoria di Perron-Frobenius alle mappe nonlineari. Nel passaggio da mappe lineari a quelle nonlineari, la teoria di Perron-Frobenius considera mappe monotone piuttosto che matrici non negative. Il contributo originale è triplice. Anzitutto, viene svolta un'analisi di convergenza del comportamento iterativo di alcune classi di mappe monotone nonlineari, stabilendo così condizioni sufficienti che garantiscono la convergenza verso un punto fisso della mappa: le matrici non negative e stocastiche per righe e non negative ne risultano essere un caso speciale. In secondo luogo, questi risultati sono impiegati in sistemi multi-agente, sia a tempo discreto che continuo: si identificano le proprietà locali della dinamica di ciascun agente affinché il sistema globale interconnesso rientri in una delle classi sopracitate, garandendone in questo modo la stabilità globale. Infine, si ricava una condizione sufficiente sulla connettività della rete di communicazione per limitare l'insieme dei punti di equilibrio ai punti di consenso, assicurando così che gli agenti raggiungano il consenso. Questi risultati non si basano su metodi standard (ad esempio, la teoria di Lyapunov) e dunque costituiscono un nuovo approccio all'analisi e al controllo di sistemi dinamici multi-agente.

Nella seconda parte, ci si concentra sulla progettazione di algoritmi di stima dinamica in reti estese che permettono di risolvere problemi specifici. Il primo problema consiste nel rompere l'effetto di sincronizzazione in reti di oscillatori armonici accoppiati in modo diffusivo. Viene sviluppata una metodologia per il progetto di una legge di retroazione locale dello stato che consente la desincronizzazione in reti connesse con interazioni arbitrarie non dirette. L'azione di controllo proposta è ottenuta tramite un nuovo protocollo per la stima distribuita del vettore Fiedler della matrice Laplaciana. Il secondo problema consiste nella stima del numero di agenti attivi in reti in cui gli agenti possono entrare e uscire. La strategia adottata consiste nella stima distribuita e dinamica del massimo tra i numeri generati localmente dagli agenti attivi e la conseguente inferenza del numero di agenti che hanno preso parte all'esperimento. Sono proposti e caratterizzati due protocolli per risolvere il problema del consenso sul valore di massimo variabile nel tempo. Il terzo problema consiste nella stima del valor medio degli stati di un'ampia rete di agenti quando solo pochi stati degli agenti sono accessibili ad un osservatore centralizzato. La strategia proposta proietta le dinamiche del sistema originale in uno spazio degli stati di dimensione inferiore, rivelandosi particolarmente utile quando si ha a che fare con reti ad ampia scala. Sono identificate le condizioni necessarie e sufficienti per l'esistenza di un osservatori lineari e sliding-mode, insieme a una caratterizzazione del loro progetto e delle proprietà di convergenza.

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I owe everything to my parents, ***Mafalda*** *and* ***Flavio****. They provided me inexhaustible and unconditional support and encouragement throughout my lifetime. I hope to manage to give them back at least a drop of the love they gave me.*

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Research is challenging, but search for love is more ambitious. I have found it in ***Silvia****, the best discovery I have ever and will ever make. Always.*

Contents

Introduction | 1

1.1 Multi-agent systems (MASs) in real life

All of us, at least once in a life time, have played board games. My favorite are cooperative games, in which a common goal must be achieved by the players whose exchange of information and knowledge of the circumstances are usually constrained by the rules. During a game, each player can only gather partial information about the other players and takes individual decisions in order to achieve the common goal, determining in this way the evolution of the game as an emergent global behavior. Two fundamental questions arise from this familiar example:

- Which is the best individual strategy to adopt?
- Is it possible to predict the evolution of the game?

Differently from game theory [182], in which each player, or agent, makes choices trying to optimize its outcome regardless of the final outcome of the other agents, this thesis focuses on the analysis and control of the emergent behavior of a global interconnected system through the design of each agent dynamics [176]. As the number of agents increases and the structure of the communications gets complicated, the flow of information and the impact of agents' behavior become less amenable to analysis, prediction and control.

[182] Parsons and Wooldridge (2002), 'Game theory and decision theory in multi-agent systems'.

[176] Olfati-Saber et al. (2007), 'Consensus and cooperation in networked multi-agent systems'.

It is quite intuitive that, such a scenario, where a large number of interacting units with limited sensing and communication capability must be individually controlled with the aim of achieving a global common goal, encompasses several real world situations. In contrast, the problem would be much easier, albeit more unrealistic and more expensive in terms of cost and computation complexity, if one of the units had full information and control on the others. From a control system perspective, the former case is much more relevant yet challenging. In fact, the design of control systems is experiencing a shift from centralized approaches, where decisions are taken and spread over the network by a centralized controller gathering the full available information, to decentralized approaches, where decisions are taken by the single units only having partial information gathered from neighboring units without a centralized controller.

Collective emergent behaviors were first observed in biology and physics [124, 229]. In biology, explaining the ability of animals that move together in a group to make collective decisions requires an understanding of the information flow mechanisms [136, 194]. In groups such as fish schools, sheep flocks or large insect swarms, usually individuals can only sense the relative motion of neighbors and are not able to distinguish one individual from another [46]. Moreover, the emergent behavior is robust to individual faults and consequently we are increasingly acknowledging that an emergent intelligence is more reliable than the intelligence provided by a few leaders [123]. In statistical physics, the study of the emergent behavior the self-propelled particles, which are described as autonomous agents converting energy from the environment into directed or persistent motion, has received a significant amount of attention [137, 218, 228]. To understand the ubiquity of such phenomena, physicists have developed a number of local interaction rules that produce flocking or swirling behaviors, thus showing that self-propelled particles share certain properties at the group level, regardless of the type of particles in the swarm.

Inspired by these biological and physical phenomena, scientists from different fields of science have developed several models to explain these behaviors and have successfully applied them to their case of study. In sociology [55, 86, 190], the opinion of the people have their own nature and their evolution is influenced by the opinion of the other people to which they have interaction, family, friends, celebrities and so on. The diversity or agreement of people's opinion, which can be also interpreted as a repeated Nash game [93][37] determining the emergent behavior of the society, is nowadays strongly altered from the advent of modern communication technologies and social life. In modern computing [27, 165, 222], the computing devices are considered as active entities capable to perceive their outer environment and acting on it, and their individual behaviors result in a behavior accountable as a solution of a given problem. Unlike in the past, this scenario is today a reality due to advances in computing and communication and a great reduction in cost of wireless devices. In robotic networks [70, 110], several robots must be coordinated in order to achieve a common task while avoiding collisions and optimizing the time spent to achieve the task. Several other real world scenarios can be described with the same line of thinking.

The so called Multi-Agent paradigm has been developed to study the above described class systems. The individuals, units or parts composing the system are called agents, and their dynamics is

[124] Krause et al. (2002), *Living in groups*.

[229] Vicsek et al. (1995), 'Novel type of phase transition in a system of self-driven particles'.

[194] Reynolds (1987), 'Flocks, herds and schools: A distributed behavioral model'.

[136] Leonard et al. (2012), 'Decision versus compromise for animal groups in motion'.

[46] Couzin et al. (2005), 'Effective leadership and decision-making in animal groups on the move'.

[123] Krause et al. (2010), 'Swarm intelligence in animals and humans'.

[218] Toner and Tu (1998), 'Flocks, herds, and schools: A quantitative theory of flocking'.

[137] Levine et al. (2000), 'Self-organization in systems of self-propelled particles'.

[228] Vicsek and Zafeiris (2012), 'Collective motion'.

[55] DeGroot (1974), 'Reaching a consensus'.

[86] Friedkin and Johnsen (1999), 'Social Influence Networks and Opinion Change'.

[190] Proskurnikov and Tempo (2017), 'A tutorial on modeling and analysis of dynamic social networks. Part I'.

[37] Cenedese et al. (2020), 'Asynchronous and time-varying proximal type dynamics in multi-agent network games'.

assumed to be thoroughly described by input/state maps. The interconnections among the agents describe a network with a specific topology. As discussed above, one of the most relevant objectives is to infer the collective emergent behavior of the network of agents from the knowledge of their individual dynamics, their coupling dynamics with other neighboring agents and the underlying network topology. In particular, the *consensus* or *synchronization* behavior has attracted considerable interest from the control community. The consensus or synchronization problem consists in the design of local interaction rules between the agents such that as global emergent behavior the network converges to a common state, which is usually called the *agreement* or *consensus state*. The desired consensus state is usually a function of either the initial or current network state, differentiating between static and dynamic consensus. A *consensus algorithm*, or *protocol*, is a local interaction rule that specifies the information exchange between an agent and its neighbors in the network.

[222] Tsitsiklis (1984), *Problems in decentralized decision making and computation.*

[27] Bertsekas and Tsitsiklis (1989), *Parallel and distributed computation: numerical methods.*

[165] Nedic et al. (2010), 'Constrained consensus and optimization in multi-agent networks'.

[110] Jadbabaie et al. (2003), 'Coordination of groups of mobile autonomous agents using nearest neighbor rules'.

[70] Fax and Murray (2004), 'Information flow and cooperative control of vehicle formations'.

After an initial period of modeling and simulation of consensus phenomena observed in biology and physics [194, 229], the control systems community become interested in this problem thanks to [110]. Here, a theoretical explanation for the consensus behavior of Vicsek model by applying graph theory and matrix theory is provided. Soon after, a cohesive overview of the basic theory, formulation and applications of consensus problems in networks of first-order integrator is given in [176]. Finally, the consensus problem of the second-order integrator MASs has been addressed in [193], where the relevance of the communication topology for achieving asymptotic agreement was emphasized, identifying the necessary condition of the presence of a globally reachable node. Nowadays, the developing of matrix and graph theory has led the study of consensus to a new phase, where the researchers focus on the design of consensus protocols under different sets of assumptions on the agents' dynamics.

[194] Reynolds (1987), 'Flocks, herds and schools: A distributed behavioral model'.

[229] Vicsek et al. (1995), 'Novel type of phase transition in a system of self-driven particles'.

[110] Jadbabaie et al. (2003), 'Coordination of groups of mobile autonomous agents using nearest neighbor rules'.

[176] Olfati-Saber et al. (2007), 'Consensus and cooperation in networked multi-agent systems'.

[193] Ren and Beard (2005), 'Consensus seeking in multiagent systems under dynamically changing interaction topologies'.

The research on consensus has witnessed a huge effort to relax the set of assumptions from various perspectives: switching network topology and delays in the communications [162, 236], presence of one or multiple leaders in the network [168], sampled-data and event based transmission [96], quantized consensus [114], clustered consensus [243], and many others. Meanwhile, an increasingly wider class of scientific and engineering problems were addressed including problems such as distributed consensus on specific functions, load balancing on networks, cooperative rendezvous in a network of mobile vehicles, synchronization of oscillators, distributed clock synchronization, and many others.

[236] Xiao and Wang (2008), 'Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays'.

[162] Munz et al. (2011), 'Consensus in multi-agent systems with coupling delays and switching topology'.

[168] Ni and Cheng (2010), 'Leader-following consensus of multi-agent systems under fixed and switching topologies'.

[96] Guo et al. (2014), 'A distributed event-triggered transmission strategy for sampled-data consensus of multi-agent systems'.

[114] Kashyap et al. (2007), 'Quantized consensus'.

[243] Yu and Wang (2010), 'Group consensus in multi-agent systems with switching topologies and communication delays'.

1.2 Topics of the thesis and outline

In this first chapter we have introduced the reader to the multi-agent paradigm which is useful to describe different real-world scenarios, while in this section we provide a brief outline of the main problems arising in the study of MASs which are addressed in this thesis. These problems are introduced and formalized in the next **Chapter 2**, along with a brief presentation of our original results and their framing in the actual literature. The thesis consists of two parts.

First part

The first part of the thesis focuses on the application of recent new advances in nonlinear *Perron-Frobenius theory* to MASs, both in continuous and discrete time.

In linear algebra, the Perron–Frobenius theory has immediate consequences on the exponential growth rate of the matrix powers which results to be controlled by the eigenvalue with the largest absolute value. In particular, the Perron–Frobenius theory describes the properties of the leading eigenvalue when the entries of the matrix are nonnegative, determining in this way the limit (infinity, finite or zero) of the power of a matrix as the exponent increases. This theory has important applications in several fields, such as robot coordination, power control in wireless networks, commodity pricing models in economics, population growth models, to name a few, including the theory of linear dynamical systems ruled by nonnegative matrices. If "classical" Perron-Frobenius theory deals with nonnegative matrices, nonlinear Perron-Frobenius theory deals with *positive* and *order-preserving* maps[1]. In broad terms, a map acting on a vector space is said to be positive if it leaves the cone of positive vectors invariant and is said to be order-preserving if it keeps ordered the images of two ordered vectors. For linear maps, positivity and order-preservation corresponds to mappings defined by nonnegative matrices, the object of study of Perron-Frobenius theory. This equivalence does not hold for nonlinear maps, thus nonlinear Perron-Frobenius theory considers maps that are both positive and order-preserving. Since the aim is to

1: See Chapter 3 and in particular Section 3.2

exploit such a theory in nonlinear dynamical systems, positive and order-preserving maps are considered.

However, in the shift from the linear to the nonlinear framework, the controlled convergence behavior of the map iteration fails. Therefore, the need to identify a more specific class of maps arises naturally, which is the first main contribution of this part: we identify a stricter version of order-preservation, which is termed as *type-K order-preservation*[2] in our previous works [58, 61], whose iterative behavior is more constrained and it allows to avoid periodic and divergent trajectories: nonnegative matrices turns out to be a special case of this class of nonlinear maps. The second main contribution is the application of this new mathematical tool to the analysis of MASs, both in discrete and continuous-time: we identify sufficient conditions on the local dynamics of the single agent ensuring that the global interconnected system falls in the class of systems under consideration. These tools do not rely on standard methods (e.g., Lyapunov theory) and thus they represent a novel approach to the analysis and control of dynamical systems and MASs. Part of the results presented in this first part can be found in [58, 61], while some results are still unpublished. The first part of the thesis consists of three chapters:

2: See Definition 3.2.3 in Section 3.2.

[58] Deplano et al. (2018), 'Lyapunov-Free Analysis for Consensus of Nonlinear Discrete- Time Multi-Agent Systems'.

[61] Deplano et al. (2020), 'A nonlinear Perron–Frobenius approach for stability and consensus of discrete-time multi-agent systems'.

- In **Chaper 3** the fundamental definitions and results of interest of nonlinear Perron-Frobenius theory are presented. In particular, the properties of order-preservation and homogeneity of a map, object of nonlinear Perron-Frobenius theory, are introduced along with their variations. The main contribution of the chapter is the identification of a specific variation of the order-preservation property, termed type-K, which constraints the dynamics of the iterative behavior of the map allowing to establish its convergence to a fixed point, if one exists. Moreover, such a property is shown to be identifiable from the sign structure of the map's Jacobian, paving the way for the application of this novel theory to MASs.
- In **Chapter 4** results of Chapter 3 are exploited to study stability and, in addition, convergence to a consensus state for two classes of discrete-time MASs where the agents evolve with nonlinear dynamics, possibly different for each agent. In particular, the classes of MASs of interest have global dynamics represented by type-K order-preserving maps possessing an extra homogeneity condition, alongside an additional connectivity condition on the topology of the network. These results generalize results that apply to linear

MASs to the nonlinear case by exploiting nonlinear Perron-Frobenius theory. Two examples of application are provided to corroborate the theoretical analysis: the first considers the susceptible-infected-susceptible (SIS) model and the second is a novel protocol to solve the max-consensus problem.

- In **Chapter 5** stability and, in addition, convergence to a consensus state analysis for a class of continuous-time MASs is carried out. The class of MASs considered in this chapter can be regarded as the continuous-time counterpart of MASs considered in Chapter 4. In particular, the focus is on the class of MASs whose global dynamics is represented by monotone and translation invariant vector fields. As a first application we consider consensus in MASs with input saturation, which may cause undesirable effects such as a performance degradation and an instability. We provide the design of a general saturating function which does not invalidate the convergence to consensus properties of the standard consensus protocol in networks of single-integrator; as a special case is derived the discontinuous control protocol. As a second application we consider the synchronization of a general class of oscillators which includes, among the others, Kuramoto oscillators and monotone oscillators. We provide sufficient conditions on the coupling functions among the oscillators, which can be directed and different for each couple of oscillators, guaranteeing local asymptotic stability and local instability of phase-synchronized solutions; by means of this result, a specific design of the coupling function is provided to establish global asymptotic stability to the synchronized state.

Second part

The second part of the thesis focuses on different dynamic estimation problems in large networks which enables to solve specific problems, namely the desynchronization in harmonic oscillator networks, the estimation of the size of an anonymous network and the design of average observers.

Synchronization of flashing fireflies, coordinated oscillations of the central pattern generator (CPG) in animal locomotion, synchronization of rotor dynamics of generators in power networks are just few examples where coupled oscillators can be found. Feedback control theories to achieve synchronization of coupled oscillators have received a great amount of interest from the researchers, while a less consideration has been given on the problem of breaking synchronization, or, equivalently, to achieving desynchronization, which can be useful in several context, e.g., to avoid pathological synchronization phenomena of neural oscillations in Parkinson's disease or reducing the stress in networked mechanical systems. In particular, in this thesis the desynchronization problem for a network of diffusively coupled harmonic oscillators has been addressed. While synchronization has been formally and easily defined as the condition maximizing the order-parameter (magnitude of the centroid of the oscillators), here the dual definition of desynchronization as the condition zeroing the order-parameter is considered. Based on this definition, a local state feedback that achieves desynchronization on harmonic oscillator networks with arbitrary undirected interaction is provided; the control action is obtained via a novel local control protocol for the distributed estimation of the Fielder vector of the Laplacian matrix.

In distributed computation, the collaborating agents need to preserve some properties and working conditions of the network, and potentially perform opportune corrective actions. In this respect, counting the number of active agents in a decentralized network is a key function, and it is crucial for topological change detection or automatic network reconfiguration. The importance of the network size estimation problem is evident from the abundance of literature on the topic and the several applications. The specific instance of the size estimation problem considered in this thesis assumes the network to be time-varying, meaning that the agent may leave or join the network, and also assumes the anonymity of the agents, meaning that the agents' IDs are not share with the neighbors. The strategy adopted in this theses takes inspiration from [225] and it is based on statistical inference concepts whose paradigm is the following: agents joining the network locally generate random numbers, distributedly compute the maximum of these generated data, finally locally compute the Maximum Likelihood estimate of the network size exploiting its probabilistic dependencies. This strategy requires a protocol to solve the dynamic max-consensus problem, which is missing in the literature. Therefore, motivated by this problem and by the lack of such a protocol in the current literature, in this thesis two novel protocols are proposed to solve the dynamic max-consensus problem which are then applied and characterized for the size estimation problem in time-varying and anonymous networks.

[225] Varagnolo et al. (2014), 'Distributed Cardinality Estimation in Anonymous Networks'.

State estimation for monitoring large-scale systems requires tremendous amounts of computational and sensing resources, which is impractical in most applications. However, knowledge of some aggregated quantity of the state suffices in several applications. Processes over physical networks such as traffic, epidemic spread and thermal control are examples of large-scale systems. Due to the diffusive nature of these systems, the average state is usually sufficient for monitoring purposes. Therefore, in this thesis we address the problem of designing average state observers of the network when only a few measurements are available. The proposed strategy considers a projected system which is obtained by projecting the dynamics of the original system to a lower dimensional state space. Such an approach is useful to deal with the complexity of large-scale systems. Necessary and sufficient conditions for the existence of a linear and a sliding mode observers are derived, along with a characterization of their design and convergence properties.

These three problems are considered in separate chapters:

- In **Chapter 6** a novel continuous-time distributed control protocol is proposed to drive the value of the state variables of a network toward the Fielder vector, up to a scale factor, assuming known algebraic connectivity. The protocol is unbiased and robust with respect to the initial network state and does not require initialization of state variables to particular values. By exploiting the proposed control protocol, a local state feedback is designed that achieves desynchronization on arbitrary undirected connected networks of diffusively coupled harmonic oscillators. The results of this chapter can be found in [62].
- In **Chapter 7** two novel consensus protocols for discrete-time MASs (MAS) are proposed to solve the dynamic consensus problem on the min/max value, i.e., the dynamic min/max-consensus problem. The absolute tracking error of the proposed distributed control protocols is theoretically characterized and it is shown to be bounded. Moreover, by tuning its parameters it is possible to trade-off convergence time for steady-state error. The proposed protocol is then applied to solve the distributed size estimation problem in a dynamic setting where the size of the network is time-varying during the execution of the estimation algorithm. The results of this chapter are still unpublished and they are available on arXiv [60], while some preliminary results were provided in [59].
- In **Chapter 8** a necessary and sufficient condition for the existence of a average state observer for large-scale linear

[62] Deplano et al. (2020), 'Distributed Fiedler Vector Estimation With Application to Desynchronization of Harmonic Oscillator Networks'.

[60] Deplano et al. (2020), 'Dynamic Min and Max Consensus and SizeEstimation of Anonymous Multi-Agent Networks'.

[59] Deplano et al. (2019), 'Discrete-Time Dynamic Consensus on the Max Value'.

systems is derived. Two design procedures are proposed: a linear observer and a sliding mode observer. When the necessary and sufficient condition is not satisfied, a further constraint design is devised to obtain an optimal asymptotic estimate of the average state in terms of minimal estimation error. In particular, the estimation problem is addressed by aggregating the unmeasured states of the large-scale system and obtaining a projected system of reduced dimension. This approach reduces the complexity of the estimation task and yields observers of reduced dimension. Moreover, it turns out that the original dimension of the system also does not affect the upper bound on the estimation error and the complexity of the observers. Part of the results presented in this chapter can be found in [169], while some are unpublished results.

[169] Niazi et al. (2019), 'Scale-free estimation of the average state in large-scale systems'.

State of the art, problems, and results

2

2.1 Multi-agent modeling of dynamical systems

A dynamical system is an abstract concept used to denote a physical system which can be described in terms of signals evolving in time. A dynamical system, is characterized by a mathematical model describing the laws governing the evolution of the signals of interest; we consider state-space representations as in Figure 2.1, which model physical systems as a set of *state variables* and *input/output* signals related by first-order differential equations (continuous-time) or difference equations (discrete-time). We use the term dynamical system to refer to either continuous-time or discrete-time dynamical systems.

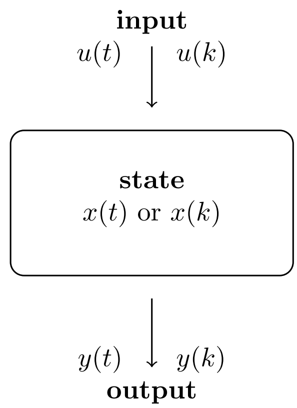

Figure 2.1: State-space representation of a dynamical system.

Every physical system is a *causal* dynamical systems, i.e., the *output* $y(t_0)$ at time t_0 depends on past and current *inputs* $u(t)$ with $t \leq t_0$, but not on future input. Thus, this text studies only causal systems. Theoretically, the inputs should be known for times back to minus infinity, which is very inconvenient, if not impossible. The concept of *state* deals with this problem: the state $x(t_0)$ of a system at time t_0 is the information at t_0 that, together with the input $u(t)$ determines the output $y(t_0)$ for all $t \geq t_0$. Knowing the state at time t_0 is equivalent to know the input applied before t_0 in determining the output after t_0.

Consider to be known the state $x(0) \in X$ of a system at the initial time $t = 0$, where X is the set of all possible states of the system. Denoting with f the law describing the evolution of the state in time, a dynamical system evolving in discrete-time reads

$$x(k+1) = f(x(k), u(k)), \quad k \in \mathbb{N}, \tag{2.1}$$

where $f : X \longrightarrow X$ is a map and the k-th iterate of map f is the n-fold composition $f^n = f \circ \ldots \circ f$, and thus the system can also be written as $x(k) = f^k(x(0))$.

Similarly, a dynamical system evolving in continuous-time reads

$$\dot{x}(t) = f(x(t), u(t)), \quad t \in \mathbb{R} \tag{2.2}$$

which identifies the one parameter family of maps $\{\varphi(t,x) : X \to X \text{ for } t \in \mathbb{R}\}$, solutions to the initial value problem. Note that for a fixed $T \in \mathbb{N}$, the iterates $x(k) = \varphi^k(T, x(0))$ form a discrete-time dynamical system for any $T \in \mathbb{R}$.

A MAS consists of a set of dynamical systems whose state evolution is interdependent, and each system is usually referred to as an agent. Consider n agents whose states are x_i with $i = 1, \dots, n$ and whose dynamics in discrete or continuous-time, respectively, is given by

$$x_i(k+1) = f_i(x(k), u_i(k)), \quad \text{or} \quad \dot{x}_i(t) = f_i(x(t), u_i(t)),$$

where u_i denotes the input of the i-th agent. By using notation from algebraic graph theory (see Appendix A.3), we model the pattern of interactions among the agents with a *graph*.

Definition 2.1.1 *A* directed *graph is a pair* $\mathcal{G} = (V, E)$*, where* $V \subset \mathbb{N}$ *is the set of* nodes *and* $E \subseteq V \times V$ *is the set of* edges*.*

A graph can be graphically represented by numbered circles denoting the nodes $i \in V$ and by arrows drawn from circle i to the circle j denoting the edges (i, j); as an example of graphical representation, a directed circle graph is drawn in Fig. 2.2. A directed graph is said to be *undirected* if for each edge $(i, j) \in E$ there also exist an edge with opposite direction $(j, i) \in E$. In Fig. 2.3 is shown an undirected star graph, where the edges are depicted as lines instead of arrows due their undirected nature. If not mentioned, a graph refers to a directed graph.

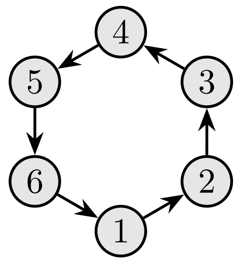

Figure 2.2: Directed cycle graph with 6 nodes: $V = \{1,2,3,4,5,6\}$ and $E = \{(1,2),(2,3),(3,4),(5,6),(6,1)\}$.

The nodes of the graph $i \in \{1, \dots, n\} = V$ represent the agents and the edges $(i, j) \in V \times V = E$ represent the existing interaction among agents i and j. In a natural way the set of neighbors is defined as $\mathcal{N}_i = \{j : (i, j) \in E\}$, which contains the agents interacting with agent i. Therefore, the dynamics $f_i(\cdot)$ of each agent can be decomposed into its own dynamics $g_i(\cdot)$, if any, and a local interaction protocol with neighboring agents $h_i(\cdot)$, as follows

$$f_i(x, u) = g_i(x_i) + h_i(u_i, x_i, x_j : j \in \mathcal{N}_i). \tag{2.3}$$

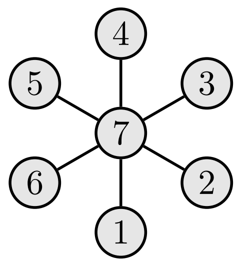

Figure 2.3: Undirected star graph with 7 nodes: $V = \{1,2,3,4,5,6,7\}$ and $E = \{(1,7),(7,1),(2,7),(7,2),(3,7),(7,3),(4,7),(7,4),(5,7),(7,5),(6,7),(7,6)\}$

Such distinction is not always made explicit and sometimes $f_i(\cdot)$ directly denotes the local interaction rule of the i-th agent. Clearly, denoting with $x = [x_1, \dots, x_2] \in X^n$ the whole state of the MAS and with $f = [f_1, \dots, f_n] : X^n \to X^n$, the MAS can be written as a dynamical system as in eq. (2.1) or eq. (2.2).

2.2 The consensus problem

An autonomous[1] MAS is completely defined by the agents' dynamics $g_i(\cdot)$, their local interaction protocol $h_i(\cdot)$ and the network topology $\mathcal{G}$. In the remainder of this section we consider a vector state space $X \subseteq \mathbb{R}$.

1: A dynamical system is said to be autonomous if it does not have any input, i.e., $u = 0$

Problem 2.2.1 Consider an autonomous MAS defined by the network topology $\mathcal{G} = (V, E)$ with agents' dynamics as in eq. (2.3). The consensus problem consists in the design of a set of local interaction protocols $h_i(\cdot)$ such that each agent converge to the same state, i.e.,

$$\lim_{t\to\infty} \left\| x_i(t) - x_j(t) \right\| = 0, \qquad \forall i, j \in V. \tag{2.4}$$

The state reached by the agent goes by the name of *agreement* or *agreement state*, where indeed the value of each agent's state is the same. Sometimes, the consensus state is required to be a specific function of the initial network state, such as the average value, the median value, the min/max value, the variance, and many others. In the next subsections we review the most popular linear protocols to achieve consensus both in continuous and discrete-time, then we briefly comment some of our related main results, which are their nonlinear counterpart. These results are discussed at length in Chapters 4-5 and relies on the theory developed in Chapter 3.

Consensus in discrete-time

Let $\mathcal{G}$ be a graph that describes the topology of the network of agents and assume that each agent is a single discrete-time integrator with dynamics

$$x_i(k+1) = x(k) + h_i(x_i(k), x_j(k) : j \in \mathcal{N}_i), \tag{2.5}$$

where $x_i \in \mathbb{R}$. One of the most popular linear local interaction protocol solving the consensus problem is provided in [175] and reads as

$$h_i(\cdot) = \varepsilon \sum_{j\in\mathcal{N}_i} \left(x_j(k) - x_i(k)\right), \tag{2.6}$$

[175] Olfati-Saber and Murray (2004), 'Consensus problems in networks of agents with switching topology and time-delays'.

The peculiarity of this protocol is its dependence from the parameter $\varepsilon \in \mathbb{R}$ and its relation to the *Laplacian* matrix L of graph $\mathcal{G}$; in fact the global multi-agent system dynamics is linear and can be written in compact form as

$$x(k+1) = \underbrace{(I - \varepsilon L)}_{P}\, x(k). \tag{2.7}$$

Definition 2.2.1 *Given a directed graph $\mathcal{G} = (E, V)$, its $|V| \times |V|$ Laplacian matrix L with elements ℓ_{ij}, $i, j \in V$, is defined by*

$$\ell_{ij} = \begin{cases} -1 & \text{if } (i,j) \in E \\ |\mathcal{N}_i| & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

The convergence properties of system (2.5)-(2.6) depends on the parameter ε and the topological structure of $\mathcal{G}$ (see Theorem 5.1 in [32]); in particular, the system reaches consensus if

1. The parameter ε satisfies $\varepsilon < \max_{i \in V} |\mathcal{N}_i|^{-1}$;
2. The graph $\mathcal{G}$ contains a globally reachable node.

[32] Bullo (2018), *Lectures on Network Systems*.

These two conditions can be derived by the several interesting properties of the Laplacian matrix. First of all, denoting with $\mathbb{1}$ and $\mathbb{0}$ the vectors of ones and zeros, respectively, it holds that $L\mathbb{1} = \mathbb{0}$. Moreover, the Laplacian matrix has always a zero eigenvalue, while all others have strictly positive real part which is less than $2\varepsilon \max_{i\in V} |\mathcal{N}_i|$. It is straightforward to notice that by construction matrices P and L share the same set of eigenvectors and that the eigenvalues μ_i of the matrix P are linked to those λ_i of L by the following $\mu_i = 1 - \varepsilon\lambda_i$. Thus, system (2.7) have a structural eigenvalue $\mu_1 = 1$ associated to the eigenvector $\mathbb{1}$. Moreover, if condition 1. is satisfied, then all eigenvalues have real part strictly less than 1 and if condition 2. is satisfied, then the eigenvalue $\mu_1 = 1$ is unique and thus the system is marginally stable and all trajectories converges to the space spanned by the eigenvector $\mathbb{1}$ associated to the simple unitary eigenvalue, thus proving the achievement of a consensus state among the agents.

Many variations of protocol in eq. (2.6) have been proposed, not necessarily in a chronological order, in several applications, such as formation control in multi-vehicle systems [70, 110], the modeling of the emergent flocking behavior [228, 229], optimization algorithms [165, 221], and many others.

[70] Fax and Murray (2004), 'Information flow and cooperative control of vehicle formations'.
[110] Jadbabaie et al. (2003), 'Coordination of groups of mobile autonomous agents using nearest neighbor rules'.
[229] Vicsek et al. (1995), 'Novel type of phase transition in a system of self-driven particles'.
[228] Vicsek and Zafeiris (2012), 'Collective motion'.
[221] Tsitsiklis et al. (1986), 'Distributed asynchronous deterministic and stochastic gradient optimization algorithms'.
[165] Nedic et al. (2010), 'Constrained consensus and optimization in multi-agent networks'.

Now, consider its natural nonlinear generalization,

$$h_i(\cdot) = \varepsilon \sum_{j \in \mathcal{N}_i} h\left(x_j(k) - x_i(k)\right), \tag{2.8}$$

where the function h is a nonlinear function of the agents' state differences. As before, the convergence properties of system (2.5)-(2.8) depends on the parameter ε, the coupling function $h(\cdot)$ and the topological structure of $\mathcal{G}$; in particular, the system reaches consensus if

1. The parameter ε satisfies $\varepsilon < \max_{i \in V} \left[|\mathcal{N}_i| \frac{\partial}{\partial x_i} h(\cdot) \right]^{-1}$;
2. The graph $\mathcal{G}$ contains a globally reachable node;
3. The function $h(\cdot)$ satisfies $h(\mathbb{0}) = 0$ and $h'(\cdot) \geq 0$;

This result is an original contribution of this thesis and it is a special case of the results provided in Chapter 4. It is worth mentioning that this results does not fall in the general convex analysis of Moreau [160], even if most of the results presented in the literature do. It is clear that if the function $h(\cdot)$ is taken as the identity map $h(x) = x$, then protocol in eq. (2.8) reduces to the one in eq. (2.6). Due to condition 3., this protocol makes the global dynamical system order-preserving and homogeneous [4, 5, 91]. We refer the reader to Section 4.1 for a detailed study of the related literature.

[160] Moreau (2005), 'Stability of multiagent systems with time-dependent communication links'.

Consensus in continuous-time

Let $\mathcal{G}$ be a graph that describes the topology of the network of agents and assume that each agent is a single continuous-time integrator with dynamics

$$\dot{x}_i(t) = h_i(x_i(t), x_j(t) : j \in \mathcal{N}_i), \tag{2.9}$$

where $x_i \in \mathbb{R}$. One of the most popular linear local interaction protocol solving the consensus problem reads as

$$h_i(\cdot) = \sum_{j \in \mathcal{N}_i} \left(x_j(t) - x_i(t)\right), \tag{2.10}$$

The global system dynamics is thus ruled by the Laplacian matrix L encoding the network topology $\mathcal{G}$, in fact it can be compactly represented by

$$\dot{x} = -Lx$$

The convergence of the system (2.9)-(2.10) requires that the topological structure of $\mathcal{G}$ contains a globally reachable node, see Theorem 7.4 in [32].

[32] Bullo (2018), *Lectures on Network Systems*.

Now, consider its natural nonlinear generalization,

$$h_i(\cdot) = \sum_{j \in \mathcal{N}_i} h\left(x_j(t) - x_i(t)\right), \tag{2.11}$$

where the function h is a nonlinear function of the agents' state differences. The convergence properties of system (2.9)-(2.11) depends on the coupling function $h(\cdot)$ and the topological structure of $\mathcal{G}$; in particular, while the graph is still required to contain a globally reachable node, the function $h(\cdot)$ must satisfy the following

$$h(\mathbb{0}) = 0, \qquad h'(\cdot) \geq 0 \tag{2.12}$$

This result is an original contribution of this thesis and it is a special case of the results provided in Chapter 5. A similar result is given in [253], where in addition the vector field of the global system is required to met an extra strict subtangentiality condition. It is clear that if the function $h(\cdot)$ is taken as the identity map $h(x) = x$, then protocol in eq. (2.11) reduces to the one in eq. (2.10). Due to condition in eq. (2.12), this protocol makes the global dynamical system monotone and translation invariant [14, 108, 210]. We refer the reader to Section 5.1 for a detailed study of the related literature.

[253] Zhiyun et al. (2007), 'State Agreement for Continuous-Time Coupled Nonlinear Systems'.

2.3 The dynamic consensus problem

The asymptotic consensus problem presented in the previous section can be recast as a problem of estimating some functions of static agents' inputs by requiring that all the agents initialize their state to their own inputs. However, the nature of decentralized control requires coordination among agents in a dynamic environment, making consensus protocols on static inputs insufficient and the need of consensus protocols tracking time-varying inputs necessary. Moreover, an asymptotic convergence to a common value is sometimes unfeasible, and the problem can be relaxed to an approximate consensus: the agents are required to converge in finite time to the desired function up to an bound. In the reminder of this section we only consider discrete-time MASs with agents evolving in $\mathbb{R}$, and the network graph is assumed to be undirected.

Problem 2.3.1 Consider a non-autonomous MAS defined by the network topology $\mathcal{G} = (V, E)$ with agents' dynamics as in eq. (2.3). The dynamic consensus problem consists in the design of a local interaction protocol $h_i(\cdot)$ such that each agent converges to a desired function $o(u(k))$ of the time-varying inputs up to a bound $\delta \in \mathbb{R}$, i.e., there exists a time T such that

$$||x_i(k) - o(u(k))|| \leq \delta, \qquad k \geq T, \quad \forall i, j \in V. \tag{2.13}$$

Differently from the static asymptotic consensus, in the finite-time dynamic approximate consensus the agents are not anymore required to achieve the exact same value, but to be close enough to the desired function of the time-varying inputs.

Whereas there exist several other formulations of the dynamic consensus problem and different objective functions are considered, the literature mainly focuses on the estimation of the average among the inputs [119]. The first strategy one could think is to employ a static average consensus protocol, e.g., the one in eq. (2.9)-(2.6), re-initializing the network's state to the current inputs' values at each sampling time. Clearly, this strategy is not optimal since at each sampling time it requires a centralized re-initialization step and the memory of the past actions is lost, severely affecting the tracking response. By taking a look from a frequency domain perspective, one deduces that, instead of the entering the reference signals as initial conditions, what is needed is to continuously inject the signals as inputs into the dynamical system, as follows

[119] Kia et al. (2019), 'Tutorial on Dynamic Average Consensus: The Problem, Its Applications, and the Algorithms'.

$$\begin{aligned} x_i(k+1) &= x(k) + \varepsilon \sum_{i \in \mathcal{N}_i} \left(x_j(k) - x_i(k)\right) + \Delta u(k) \\ x_i(0) &= u_i(0) \end{aligned},$$

where $\Delta u(k) = u(k+1) - u(k)$. This enables the system to respond to signals' changes without any need for re-initialization [254]. This algorithm has been derived from the original protocol proposed in continuous-time in the pioneering work [213], where the derivative $\dot{u}$ of the inputs were considered instead of the difference $\Delta u(k)$. Both algorithms require a specific initialization of the network and cannot handle noise in the communication or link failures. To overcome these limitations several approaches have been been proposed both in continuous-time [92, 117] and in discrete-time [159, 224].

[254] Zhu and Martínez (2010), 'Discrete-time dynamic average consensus'.

[213] Spanos et al. (2005), 'Dynamic consensus on mobile networks'.

On the other hand, the estimation of average value is not the only attractive goal. In fact, there is a certain number of static protocols for estimating the median value [81, 242] which have been extended to the dynamic case [51, 227], and an even more peculiar attention has been paid to the estimation of the max value both in continuous [45] and discrete-time [217], which have not been extended to the dynamic case, unfortunately. We refer the reader to Section 7.1 for a detailed study of the related literature.

[51] Dashti et al. (2019), 'Dynamic Consensus on the Median Value in Open Multi-Agent Systems'.
[227] Vasiljevic et al. (2020), 'Dynamic Median Consensus for Marine Multi-Robot Systems Using Acoustic Communication'.

Dynamic consensus on the max value

The most popular yet simple discrete-time protocol to solve the estimation of the max value of the initial state of the is the one presented in [217]. If applied to the estimation of the average of static inputs, the protocol requires that the agents first initialize their state to the value of the inputs and then update their state at each iteration by taking the maximum among the state values of the neighbors and their own state. The protocol reads as

[217] Tahbaz-Salehi and Jadbabaie (2006), 'A one-parameter family of distributed consensus algorithms with boundary: From shortest paths to mean hitting times'.

$$\begin{aligned} x_i(k+1) &= \max_{j \in \mathcal{N}_i \cup \{i\}} \{x_j(k)\} \\ x_i(0) &= u_i(0) \end{aligned} .$$

A dynamical version of this protocol which enables the agents to track the maximum of time-verying inputs and it is robust to a specific initialization of the network is the following

$$x_i(k+1) = \max_{j \in \mathcal{N}_i \cup \{i\}} \{x_j(k) - \alpha, u_i(k)\},$$

where $\alpha > 0$ is a design parameter. As in the dynamic average protocol, one needs to continuously inject the inputs into the agents dynamics. This protocol is one of the two novel protocols proposed in this thesis to solve the dynamic consensus problem on the max value and it is fully characterized in Chapter 7.

2.4 The (de)synchronization problem

Consensus and synchronization problems in networks of dynamical agents are typically solved with diffusive couplings, i.e., distributed control laws based on the differences of neighboring agents' states. In particular, synchronization in ensembles of coupled oscillators has attracted a great interest in the scientific community. Well-known examples are the classical consensus protocol in (2.6) presented in [175], its extension to harmonic oscillators [192] and

[175] Olfati-Saber and Murray (2004), 'Consensus problems in networks of agents with switching topology and time-delays'.
[192] Ren (2008), 'Synchronization of coupled harmonic oscillators with local interaction'.
[128] Kuramoto (1975), 'Self-entrainment of a population of coupled nonlinear oscillators'.

its nonlinear counterpart in Kuramoto oscillator networks [128].

An oscillator exhibits a periodic motion that repeats itself in a regular cycle, such as a cosine wave with frequency $\omega \in \mathbb{R}$,

$$y_i(t) = M_i \cos(\omega t + \varphi_i),$$

which are characterized by their amplitude M_i and their phase $\theta_i(t) = \omega t + \varphi_i$.

Phase oscillator model

If the magnitude of all oscillators is the same, without loss of generality it can be assumed to be unitary and the only state variable to be modeled is the phase $\theta_i = \omega t + \varphi_i$, whose isolated dynamics is $\dot{\theta}_i = \omega$. The interaction topology among the oscillators is modeled by a graph $\mathcal{G}$ and the coupling is considered to be diffusive

$$\dot{\theta}_i(t) = \omega + \sum_{i \in \mathcal{N}_i} h_{ij}(\theta_j(t) - \theta_i(t)). \quad (2.14)$$

Note that the discussion here is simplified to the case of identical oscillators, i.e., identical frequencies $\omega_i = \omega$ for all $i = 1, \ldots, n$.

where $h_{ij} : \mathbb{S} \to \mathbb{R}$ are 2π-periodic coupling function. A synchronization measure is the magnitude R of the so-called order parameter introduced by [128] as

$$Re^{j\phi} = \sum_{i=1}^{n} e^{j\varphi_i}.$$

which constitutes the centroid of all oscillators represented as points on the unit circle. Maximal synchronization happens if $R = 1$, i.e., when all oscillators have the same phase; minimal synchronization, or equivalently, maximal desynchronization happens if $R = 0$, i.e., when all oscillators are spaced equally on the unit circle.

Problem 2.4.1 Consider a network of n identical oscillators with dynamics as in (2.14) coupled according to graph $\mathcal{G}$. The phase synchronization problem consists in the design of a local interaction protocol $h(\cdot)$ such that for $t \to \infty$ the oscillators reach a configuration for which

$$R = 1 \Leftrightarrow \left\| \theta_i(t) - \theta_j(t) \right\| = 0, \quad (2.15)$$

i.e., the phases reaches a consensus state or, equivalently, the centroid is on the unitary circle of the Complex Plane.

The often encountered and most thoroughly studied case is that of anti-symmetric coupling without higher-order harmonics, that is, the sinusoidal coupling $h_{ij}(\cdot) = sin(\cdot)$. In particolar, the seminal work of Kuramoto [128] considered the coupled oscillator dynamics in eq. (2.14) with a complete interaction graph and uniform weights,

$$\dot{\theta}_i(t) = \omega + \frac{K}{n}\sum_{i=1}^{n} \sin(\theta_j(t) - \theta_i(t)),$$

showing by a potential landscape analysis that the network achieve phase synchronization for any value of the coupling gain K. The interest of the control community in oscillator networks was initially triggered by [111, 160], who analyzed networks of identical oscillators as nonlinear extensions of the consensus protocol in eq. (2.6). We refer the reader to [66] for a comprehensive review of the state-of-the art results on Kuramoto oscillators with several generalizations, such as heterogeneous natural frequencies, sparse network topologies and infinite oscillator populations.

[111] Jadbabaie et al. (2004), 'On the stability of the Kuramoto model of coupled nonlinear oscillators'.

[160] Moreau (2005), 'Stability of multiagent systems with time-dependent communication links'.

Most of the results presented in [66] can be extended to more general anti-symmetric and 2π-periodic coupling functions as long as the coupling is diffusive and bidirectional. However, in some applications, the coupling topology is inherently directed [148], for which there are only a few theoretical investigations. This consideration motivates the analysis carried out in Chapter 5 for the synchronization stability of oscillator networks with directed coupling and heterogeneous functions $h_{ij}(\cdot)$ for any pair (i, j) of oscillators, satisfying $h(\mathbb{0}) = 0$ and the following piecewise monotonicity condition previously considered also in [146],

[66] Dörfler and Bullo (2014), 'Synchronization in complex networks of phase oscillators: A survey'.

[146] Mallada et al. (2015), 'Distributed synchronization of heterogeneous oscillators on networks with arbitrary topology'.

$$\frac{d}{d\theta}h(\theta) = \begin{cases} > 0 & \theta \in (-\alpha, \alpha) \\ < 0 & \theta \in (-\pi, -\alpha) \cup (\alpha, \pi) \end{cases}, \qquad \alpha \in [0, \pi].$$

In Section 5.4 we show that if a phase synchronized exists, it is locally exponentially stable and that always exists a design of the parameter α ensuring global phase synchronization.

Harmonic oscillator model

If the magnitude M_i of the oscillators is not necessarily same, the system to be modeled is of second order. A model that have recently attracted increasing attention is the one of harmonic oscillators [192]. In fact, due to their theoretical and practical significance, networks of harmonic oscillators have been applied to

[192] Ren (2008), 'Synchronization of coupled harmonic oscillators with local interaction'.

address several problems electrical networks [223], in quantum electronics-mechanics-optics [28, 155, 247], resonance phenomena[208], motion coordination [21, 141] and acoustic vibrations [252].

The dynamics of harmonic oscillators with state $x_i \in \mathbb{R}^2$ reads as

$$\begin{aligned} \dot{x}_i(t) &= \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} x_i(t), + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot h_i(y_i(t), y_j(t) : j \in \mathcal{N}_i) \\ y_i(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x_i(t) \end{aligned} \tag{2.16}$$

where the interaction topology among the oscillators is modeled by a graph $\mathcal{G}$. The synchronization measure proposed in Chapter 6 is the magnitude R of the generalized order parameter

$$Re^{j\phi} = \frac{1}{\sum_{i=1}^{n} M_i} \sum_{i=1}^{n} M_i e^{j\varphi_i},$$

which constitutes the centroid of all oscillators represented as points on different circles of radius M_i. Maximal synchronization happens if $R = \sum_{i=1}^{n} M_i$, i.e., when all oscillators have the same phase; minimal synchronization, or equivalently, maximal desynchronization happens if $R = 0$, i.e., when all oscillators spaced on the complex plane such that their centroid is at the origin. This definition of (de)synchronization takes clear inspiration from the one proposed by Kuramoto.

Desynchronization has significance in several fields: treatment of neurological diseases (e.g., epilepsy and Parkinson) is addressed by means of desynchronization in neuronal networks [2, 84, 116, 156, 158, 180, 187, 189]; desynchronization is also a useful primitive for periodic resource sharing and applies to sensor network applications [44, 54, 153, 184]; finally, another important application is the motion coordination [15, 34, 90, 135].

Problem 2.4.2 Consider a network of n identical oscillators with dynamics as in (2.16) coupled according to graph $\mathcal{G}$. The desynchronization problem consists in the design of a local interaction protocol $h(\cdot)$ such that for $t \to \infty$ the oscillators reach a configuration for which

$$R = 0 \Leftrightarrow \mathbb{1}^\top y(t) = 0, \tag{2.17}$$

i.e., the collective dynamics is non-null with zero mean or, equivalently, the centroid is at the origin of the Complex Plane.

Currently, there are not studies on the desynchronization problem in networks of harmonic oscillators, even with different formulations. A first solution to this problem in the case where the oscillators are subject to a diffusive coupling is provided in Chapter 6. The proposed strategy employ the strategy to use the mean field of the oscillator network to suppress the synchronization behavior; this strategy has been employed with other oscillator models, e.g., in [2, 84].

[84] Franci et al. (2012), 'Desynchronization and inhibition of Kuramoto oscillators by scalar mean-field feedback'.
[2] Adomaitienė et al. (2019), 'Suppressing synchrony in an array of the modified FitzHugh–Nagumo oscillators by filtering the mean field'.

2.5 The observation problem

Consider a network with a very large number of agents, leading to a so-called large-scale dynamical network system, and consider the case in which the agents have not computational capabilities and/or the agents are not allowed or able to share their actual state. The monitoring of such a large-scale system would require to equip each agent with a computational unit and the permission to access the agents' state and its neighbors by the unit. This solution may be often unfeasible due to a limited number of computational unit to be deployed in the network, for instance for cost reasons, and due to a restricted access to the agents' states. This can make the system unobservable in the sense that real-time estimation of the whole network is unfeasible. It is reasonable, therefore, to monitor the network system by dynamically estimating some aggregated state profiles, i.e., some functional of the network's state.

The observation problem is thus the one of reconstructing or estimating the state or a linear combination of the states of the system using the input and output measurements [48], which can be traced back to the seminal works of Luenberger [144]. The global linear system can be compactly represented by

[48] Darouach (2000), 'Existence and design of functional observers for linear systems'.
[144] Luenberger (1971), 'An introduction to observers'.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ z(t) &= Lx(t) \end{aligned} , \tag{2.18}$$

where x, y, u, are the state, output and input of the system, respectively, and z is the quantity to be estimated. The aim is the construction of an observer of the form

$$\begin{aligned} \dot{w}(t) &= f(w(t), y(t), u(t)) \\ \hat{z}(t) &= g(w(t), y(t)) \end{aligned} , \tag{2.19}$$

where $\hat{z}$ constitutes an estimate of the desired quantity z.

Problem 2.5.1 Consider a linear network as in eq. (2.18). The observation problem consists in the design of an observer as in eq. 2.19 such that

$$\lim_{t\to\infty} ||\hat{z}(t) - z(t)|| = 0.$$

In particular, the average estimation in linear time-invariant systems has recently attracted some attention [197], which is meaningful in many applications such as urban traffic networks [42], building thermal systems [56], epidemic spread over networks [152], and power grids [179].

[197] Sadamoto et al. (2017), 'Average State Observers for Large-Scale Network Systems'.

The specific instance of the problem considered in Chapter 8 is the estimation of the average unmeasured state of the system when a few nodes $p \ll n$ of the network are measured, i.e.,

$$C = \begin{bmatrix} \mathbb{0} & I_p \end{bmatrix}, \qquad L = \frac{1}{n}\begin{bmatrix} \mathbb{1}_{n-p}^\top & \mathbb{0}_p^\top \end{bmatrix}.$$

Necessary and sufficient conditions for the existence of the observer are derived, which do not requires the computation of ranks of a concatenation of system matrices as in [71, 195], along with two different designs.

[71] Fernando et al. (2010), 'Functional observability and the design of minimum order linear functional observers'.
[195] Rotella and Zambettakis (2015), 'A note on functional observability'.

Analysis of MASs via nonlinear Perron-Frobenius theory

Stability of nonlinear dynamical systems 3

In this chapter autonomous discrete-time dynamical systems evolving in the real vector space $\mathbb{R}^n$ are considered,

$$x(k+1) = f(x(k)) \quad \text{or equivalently} \quad x(k) = f^k(x(0)), \quad (3.1)$$

where $x(0) \in \mathbb{R}^n$ is the initial condition of the system.

When the map $f(\cdot)$ is a linear operator, i.e., $f(x) = Ax$ where $A \in \mathbb{R}^{n \times n}$ is a square matrix, the classical Perron-Frobenius theory is a cornerstone in the convergence analysis of the asymptotic behavior of the system. This theory is widely established and it is briefly recalled in Section 3.1. In the past few decades a number of nonlinear extensions of Perron-Frobenius theory have been obtained, providing an extensive analysis of various classes of nonlinear maps and give information about their iterative behavior and periodic trajectories. It is natural to think how the results for linear dynamical systems would translate into the nonlinear framework.

In Section 3.1 a self-contained presentation of some basic results of classical Perron-Frobenius theory for nonnegative matrices is given along with their application to the convergence analysis of dynamical systems when the matrix is also stochastic. In Sections 3.2-3.3 the class of nonlinear maps which are the counterpart of nonnegative and stochastic matrices is introduced; these maps are shown to possess the property of non-expansivess in Section 3.4. The main result of this chapter is the convergence analysis for these classes of maps and are given in Section 3.5, while its application to the convergence of nonlinear MASs in discrete and continuous-time is provided and discussed at length in Chapters 4-5, respectively.

3.1 Background on linear PF theory

The asymptotic behavior of a discrete-time dynamical system (3.1) with linear dynamics is determined by asymptotic limit of the powers of its state transition matrix.

Contents in this section takes inspiration from the book "Lectures on Network Systems" of Bullo in [32].

Definition 3.1.1 *A matrix $A \in \mathbb{R}^{n\times n}$ is:*

(i) divergent *if* $\lim_{k\to+\infty} A^k = [\infty]$;
(ii) periodic *if* $\lim_{k\to+\infty} A^k$ *does not exists;*
(iii) semi-convergent *if* $\lim_{k\to+\infty} A^k$ *exists;*
(iv) convergent *if* $\lim_{k\to+\infty} A^k = \mathbb{0}_{n\times n}$

The next theorem recalls necessary and sufficient convergent conditions of any square matrix.

Theorem 3.1.1 *A matrix $A \in \mathbb{R}^{n\times n}$ is:*

(i) semi-convergent *if and only if $\rho(A) \leq 1$, there exists no eigenvalue of unitary norm except possibly $\lambda = 1$ and if $\lambda = 1$ is an eigenvalue then it is semisimple.*
(ii) convergent *if and only if $\rho(A) < 1$.*

$\rho(A)$ denotes the spectral radius of the matrix A, see Appendix A.2

Classical Perron-Frobenius theory is instrumental in the convergence analysis of powers of nonnegative matrices because it characterizes their spectral properties. The square matrix $A = \{a_{ij}\} \in \mathbb{R}^{n\times n}$ is

(i) *nonnegative* if $a_{ij} \geq 0$ for all $i, j \in \{1, \ldots, n\}$;
(ii) *positive* if $a_{ij} > 0$ for all $i, j \in \{1, \ldots, n\}$;

Similar definitions apply to vectors in $\mathbb{R}^n$. However, requiring that a matrix is nonnegative is not enough to guarantee its convergence, as shown in the following examples, coherently with Theorem 3.1.1:

- Matrix $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ with $\sigma(A) = \{1, 1\}$ is divergent.
- Matrix $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ with $\sigma(A) = \{0, 1\}$ is semi-convergent.
- Matrix $A_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ with $\sigma(A) = \{0, 0\}$ is convergent.
- Matrix $A_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with $\sigma(A) = \{1, -1\}$ is periodic.
- Matrix $A_5 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$ with $\sigma(A) = \{1, -1/2\}$ is semi-convergent.

$\sigma(A)$ denotes the set of eigenvalues of A, see Appendix A.2

It is therefore necessary to identify additional properties in order to differentiate among nonnegative matrices those who are semi-convergent. A square matrix $A = \{a_{ij}\} \in \mathbb{R}^{n\times n}$ with $n \geq 2$ is

(i) *irreducible* if $\sum_{k=0}^{n-1} A^k$ is positive;

(ii) *primitive* if there exists $k \in \mathbb{N}$ such that A^k is positive.

Figure 3.1 shows the relations among these properties, highlighting that positivity is the strongest and nonnegativity is the weakest. Note that the inclusions in the diagram are strict, as the following counter examples show:

- A_3 is nonnegative but not irreducible;
- A_4 is irreducible but not primitive;
- A_5 is primitive but not positive.

The following result are due to Perron and Frobenius [87–89, 185].

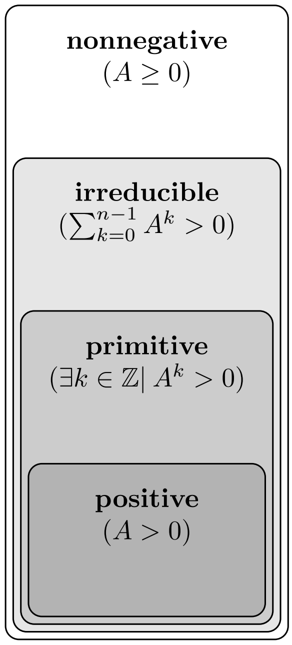

Figure 3.1: The set of nonnegative square matrices and its subsets of irreducible, primitive and positive matrices.

[185] Perron (1907), 'Zur theorie der matrices'.
[87] Frobenius (1908), *Über matrizen aus positiven elementen*.
[88] Frobenius (1908), *Über matrizen aus positiven elementen II*.
[89] Frobenius (1912), *Über matrizen aus nicht negativen Elementen*.

Theorem 3.1.2 *Let $A \in \mathbb{R}^{n \times n}$ with $n \geq 2$. If A is nonnegative, then*

(i) there exists a real eigenvalue $\lambda \geq |\mu| \geq 0$ for all other eigenvalues μ,

If additionally A is irreducible, then

(ii) the eigenvalue λ is strictly positive and simple,

If additionally A is primitive, then

(iii) the eigenvalue λ satisfies $\lambda > |\mu|$ for all other eigenvalues μ.

Given a primitive matrix A with dominant eigenvalue $\lambda = 1$, the Perron-Frobenius Theorem 3.1.2 has immediate consequences for the behavior of A^k, guaranteeing its semi-convergence by exploiting Theorem 3.1.1. Therefore, the characterization of nonnegative matrices with a unitary dominant eigenvalue has led to the identification of the class of row-stochastic matrices. A square matrix $A = \{a_{ij}\} \in \mathbb{R}^{n \times n}$ is said to be row-stochastic if it is nonnegative and if $A\mathbb{1} = \mathbb{1}$, or, equivalently, if $\sum_{j=1}^{n} a_{ij} = 1$ for all $i \in \{1, \ldots, n\}$. For row-stochastic matrices the next lemma holds.

Lemma 3.1.3 *For a row-stochastic matrix A:*

- *$\lambda = 1$ is an eigenvalue;*
- *The spectral radius is unitary, i.e, $\rho(A) = 1$.*

By combining the notion of row-stochasticity and the Perron-Frobenius Theorem 3.1.2, one gets the next theorem that is widely used in the field of network systems.

Theorem 3.1.4 *For a primitive row-stochastic matrix A,* *(i)* $\rho(A)$ *is a simple eigenvalue strictly larger than the magnitude of all other eigenvalues, hence* A *is semi-convergent;*

(ii) *the limiting matrix is* $A_\infty = \mathbb{1} w^\top$, *where* w *is the left dominant eigenvector of* A *with eigenvalue* 1 *such that* $\mathbb{1}^\top w = 1$;

(iii) *the associated dynamical system in* (3.1) *is stable and converges to an equilibrium*

$$\lim_{k\to\infty} x(k) = (w^\top x(0))\mathbb{1}.$$

In this case the system is said to achieve consensus and $w^\top x(0)$ *is the consensus value.*

In the next sections, the above theorem is generalized to nonlinear maps which possess the properties of order-preservation (instead of nonnegativity) and homogeneity (instead of row-stochasticity). These properties are introduced in Sections 3.2-3.3, respectively. In Section 3.5 a convergence analysis of their iterative behavior is provided, generalizing point (i) of Theorem 3.1.4. Finally, in Chapters 4-5 local sufficient conditions to establish convergence to a consensus state of MASs ruled by the considered class of maps are given, thus generalizing (iii) of Theorem 3.1.4.

3.2 Order-preservation of maps

Nonnegative matrices leave the cone of nonnegative vectors in $\mathbb{R}^n$ invariant, i.e., each nonnegative vector is mapped into another nonnegative vector. This is a crucial property and part of the Perron-Frobenius theory can be generalized to maps, either linear or nonlinear, that leave a cone in a vector space invariant. This important observation was made by Krein and Rutman in their pioneering work [126], in which they studied linear operators that leave a cone in a possibly infinite-dimensional normed space invariant.

[126] Krein and Rutman (1948), *Linear operators leaving invariant a cone in a Banach space*.

Instead, here the focus is on nonlinear maps acting on a real finite-dimensional vector space, leaving invariant the *standard positive cone*

$$\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, \forall i \in \{1, \dots, n\}\}. \qquad (3.2)$$

Hence, a map leaving the standard positive cone invariant is said to be *positive*, according to the following definition.

Definition 3.2.1 *A map* $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is said to be* positive *if* $f(\mathbb{R}^n_+) \subseteq \mathbb{R}^n_+$, *i.e., if map* $f(\cdot)$ *maps nonnegative vectors into nonnegative vectors.*

The standard positive cone $\mathbb{R}^n_+$ is a partially ordered set with respect to the natural order relation $\leq$. In other words, the standard positive cone $\mathbb{R}^n_+$ induces a partial ordering $\leq$ on $\mathbb{R}^n$ by

$$\begin{aligned} y \leq z \text{ or } z \geq y \quad &\Leftrightarrow \quad z - y \in \mathbb{R}^n_+, \\ y \lneq z \text{ or } z \gneq y \quad &\Leftrightarrow \quad z - y \in \mathbb{R}^n_+ \setminus \{0\}, \\ y < z \text{ or } z > y \quad &\Leftrightarrow \quad z - y \in \text{int}(\mathbb{R}^n_+), \end{aligned}$$

and so, in the sequel, the ordered real vector space $(\mathbb{R}^n, \leq)$ is considered. The partial ordering $\leq$ yields an equivalence relation $\sim$ on $\mathbb{R}^n_{\geq 0}$, i.e., x is equivalent to y ($x \sim y$) if there exist $\alpha, \beta \geq 0$ such that $x \leq \alpha y$ and $y \leq \beta x$. The equivalence classes are called *parts* of the cone of nonnegative real vectors and the set of all parts is denoted by $\mathcal{P}$. It can be shown (see [4]) that the cone $\mathbb{R}^n_{\geq 0}$ has exactly 2^n parts, which are given by

[4] Akian et al. (2006), 'Iteration of order preserving subhomogeneous maps on a cone'.

$$P_I = \{x \in \mathbb{R}^n_{\geq 0} \,|x_i > 0, \; \forall i \in I \text{ and } x_i = 0 \text{ otherwise}\},$$

with $I \subseteq \{1, \dots, n\}$. We define a partial ordering on the set of parts $\mathcal{P}$ given by $P_{I_1} \leq P_{I_2}$ if $I_1 \subseteq I_2$.

If the map $f(\cdot)$ is a linear map, the cone invariance condition $f(\mathbb{R}^n_+) \subseteq \mathbb{R}^n_+$ is equivalent to the property of *order-preservation*. Roughly speaking, an order-preserving map keeps ordered the image of two ordered vectors.

Definition 3.2.2 *A map* $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is said to be*

- order-preserving *if* $\forall x, y \in \mathbb{R}^n$ *it holds*

$$x \leq y \Rightarrow f(x) \leq f(y).$$

- strictly order-preserving *if* $\forall x, y \in \mathbb{R}^n$ *it holds*

$$x \lneq y \Rightarrow f(x) \lneq f(y).$$

- strongly order-preserving *if* $\forall x, y \in \mathbb{R}^n$ *it holds*

$$x \lneq y \Rightarrow f(x) < f(y).$$

For linear maps, order-preservation and positivity are equivalent properties and correspond to mappings defined by nonnegative matrices, the object of study of classical Perron-Frobenius theory. This equivalence does not hold for nonlinear maps, thus nonlinear Perron-Frobenius theory considers maps that are both positive and order-preserving.

In addition to these broadly known notions of order-preservation, here a less acknowledged notion that is in between strict and strong order-preservation is introduced and denoted as *type-K order-preservation*. The term *type-K* is likely related to Kamke, as it will be clear in the following. It plays a pivotal role in the characterization of the class of nonlinear systems of interest and which will be discussed at length in Chapters 4-5.

Definition 3.2.3 *A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be* type-K order-preserving *if $\forall x, y \in \mathbb{R}^n$ and $x \lneq y$ it holds*

$$(i)\ \ x_i = y_i \Rightarrow f_i(x) \le f_i(y)\,,$$
$$(ii)\ \ x_i < y_i \Rightarrow f_i(x) < f_i(y)\,,$$

for all $i = 1, \dots, n$, where f_i is the i-th component of f.

Figure 3.1 shows the relation among these properties, highlighting that positivity is the strongest and nonnegativity is the weakest. Note that the inclusions in the diagram are strict, in fact, given $x, y \in \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$, as the following counter examples show:

- $f(x, y) = [1, 1]^\intercal$ is order-preserving but not strictly order-preserving;
- $f(x, y) = [y, x]^\intercal$ is strictly order-preserving but not type-K;
- $f(x, y) = [\sqrt{x} + y, y]^\intercal$ is type-K order-preserving but not strongly order-preserving.

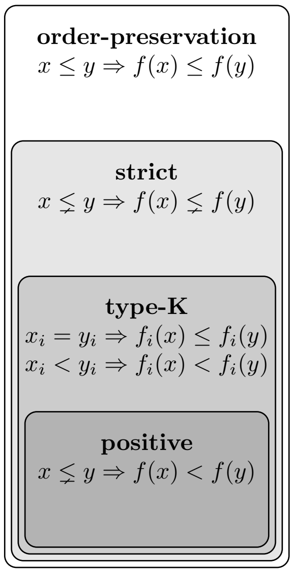

Figure 3.2: The set of order-preserving maps and its subsets of strict, type-K and strong order-preserving maps.

As it will be shown in Section 3.5, positive and type-K order-preservation plays a pivotal role in the characterization of the class of maps of interest.

To decide whether a map $f(\cdot)$ is type-K order-preserving one can use, besides the definition, the sign structure of its Jacobian matrix. This provides a practical tool to establish order-preservation of a given function.

Proposition 3.2.1 *The map $f : \mathbb{R}^n \to \mathbb{R}^n$ is type-K order-preserving if and only if its Jacobian matrix is Metzler with strictly positive diagonal elements, i.e.,*

$$\partial f_i / \partial x_i > 0 \text{ and } \partial f_i / \partial x_j \ge 0 \text{ for } i \ne j\,. \tag{3.3}$$

Proof. Let $x \in \mathbb{R}^n$ and, without lack of generality, $y = x + \varepsilon e_j$ where $\varepsilon > 0$ and e_j denotes a canonical vector with all zero values but the j-th which is 1, thus $x \lneq y$. If (5.3) holds, then

a) If $i \neq j$ then $y_i = x_i + \varepsilon 0 = x_i$ and

$$\frac{\partial f_i(x)}{\partial x_j} = \lim_{\varepsilon \to 0} \frac{f_i(x + \varepsilon e_j) - f_i(x)}{\varepsilon}.$$

Condition (i) of Definition 3.2.3 holds if and only if $f_i(x) \leq f_i(y) = f_i(x + \varepsilon e_j)$. This is equivalent to a Jacobian matrix being Metzler.

b) If $i = j$ then $y_i = x_i + \varepsilon 1 > x_i$ and

$$\frac{\partial f_i(x)}{\partial x_i} = \lim_{\varepsilon \to 0} \frac{f_i(x + \varepsilon e_i) - f_i(x)}{\varepsilon}.$$

Condition (ii) of Definition 3.2.3 holds if and only if $f_i(x) < f_i(y) = f_i(x + \varepsilon e_j)$. This is equivalent to a Jacobian matrix with strictly positive diagonal

The necessity and sufficiency of the above statements completes the proof. □

To conclude this section, an intuitive graphical representation of type-K order-preservation is given for maps acting on $\mathbb{R}^2$, highlighting in this way what is the main difference with the other notions, see Figures 3.3-3.4-3.5. Consider the real vector space $\mathbb{R}^2$ and two ordered vectors $x = [0,\ 1]^\intercal$ and $y = [0,\ 3]^\intercal$ such that $x \lneq y$. The mapping $f(x)$ identifies a cone which is highlighted in red in the figures; the bold lines represents the faces of the cone and the red faded square represents its interior. As it is shown in the figures:

- Order-preservation allows to map vector y either into the interior of the cone or into the faces;
- Strong order-preservation allows to map vector y only into the interior of the cone;
- Type-K order-preservation allows to map vector y into the interior of the cone but also in one of the two faces of the cone. In fact, points x, y do not lie in the vertical face of the cone with origin in x, thus type-K order-preservation prevents their mappings $f(x)$, $f(y)$ to lie into this face.

Consider as a simple example the map $f(x, y) = [y,\ x]^\intercal$, which is order-preserving but not type-K order-preserving. The sequence

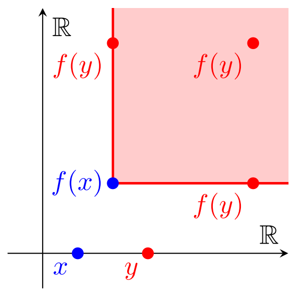

Figure 3.3: Order-preserving map

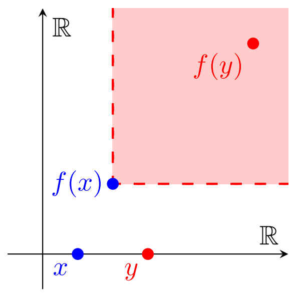

Figure 3.4: Strong order-preserving map

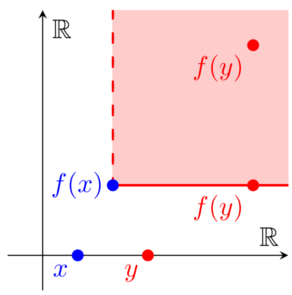

Figure 3.5: Type-K order-preserving map

of points generated by the iteration of the map starting at points $x = [0\,0]^{\mathsf{T}}$ and $y = [0,\ 1]^{\mathsf{T}}$ is shown next,

$$\begin{bmatrix}0\\0\end{bmatrix} \to \begin{bmatrix}0\\0\end{bmatrix} \to \cdots \to \begin{bmatrix}0\\0\end{bmatrix} \to \cdots$$
$$\begin{bmatrix}0\\1\end{bmatrix} \to \begin{bmatrix}1\\0\end{bmatrix} \to \cdots \to \begin{bmatrix}0\\1\end{bmatrix} \to \begin{bmatrix}1\\0\end{bmatrix} \to \cdots$$

It can be noticed that the point x is mapped into itself by $f(x)$, and thus it constitutes a *fixed point* of the map; on the contrary, the point y is mapped into itself by the map $f^2(x)$, and thus it constitutes a *periodic point* of *period* equals to 2 of the map $f(x)$. Such periodic behaviors of the iteration of a map can be avoided by the stricter type-K order-preservation assumption. In fact, as better explained in Section 3.5, positiveness and type-K order-preservation of the map $f(\cdot)$ prevents the mapping of a point lying in a face of standard positive cone to be mapped into another face (as exemplified in the previous example), thus avoiding periodicity of the sequence of points generated by the iteration of a map.

3.3 Homogeneity of maps

In nonlinear Perron–Frobenius theory one usually considers order-preserving maps satisfying an additional assumption such as homogeneity and its derivations. Next, the definition of homogeneity in the multiplicative setting is given.

Definition 3.3.1 *A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be*

- homogeneous *if $\forall x \in \mathbb{R}^n$ and $\alpha \in (0,1)$ it holds*

$$\alpha f(x) = f(\alpha x).$$

- subhomogeneous *if $\forall x \in \mathbb{R}^n$ and $\alpha \in (0,1)$ it holds*

$$\alpha f(x) \leq f(\alpha x).$$

Homogeneity implies subhomogeneity, i.e., every homogeneous map is subhomogeneous. The converse relation does not hold.

Order-preserving subhomogeneous maps arise naturally in several fields, such as the study of means, matrix scaling problems and nonlinear matrix equations (see Section 1.4 in [134] for an introduction and references).

[134] Lemmens and Nussbaum (2012), *Nonlinear Perron-Frobenius Theory*.

Consider the bijective mapping between $\mathbb{R}^n$ and $\mathrm{int}(\mathbb{R}^n_+)$ given by the exponential map $E : \mathbb{R}^n \longrightarrow \text{int}(\mathbb{R}^n_+)$ with

$$E(x) = (e^{x_1}, \ldots, e^{x_n}), \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

and its inverse, the logarithmic map $L : \text{int}(\mathbb{R}^n_+) \longrightarrow \mathbb{R}^n$ with

$$L(x) = (\ln(x_1), \ldots, \ln(x_n)), \qquad x = (x_1, \ldots, x_n) \in \text{int}(\mathbb{R}^n_+).$$

Each element of $\mathbb{R}^n$ is paired with exactly one element of $\text{int}(\mathbb{R}^n_+)$. This bijection allows one to translate these notions of homogeneity given in the standard positive cone $\mathbb{R}^n_+$ into the whole real vector space $\mathbb{R}^n$ as follows.

Definition 3.3.2 *A map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be*

- plus-homogeneous *if $\forall x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ it holds*

$$f(x + \alpha \mathbb{1}) = f(x) + \alpha \mathbb{1}.$$

- plus-subhomogeneous *if $\forall x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ it holds*

$$f(x + \alpha \mathbb{1}) \leq f(x) + \alpha \mathbb{1}.$$

Plus-homogeneity implies plus-subhomogeneity, i.e., every plus-homogeneous map is plus-suhbhomogeneous. The converse relation does not hold.

It is easy to verify that if $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a type-K order-preserving plus-(sub)homogeneous map, then the *log-exp transform* $g : \text{int}(\mathbb{R}^n_+) \longrightarrow \text{int}(\mathbb{R}^n_+)$ of $f(\cdot)$ given by

$$g(z) = (E \circ f \circ L)(z), \qquad z \in \text{int}(\mathbb{R}^n_+) \tag{3.4}$$

is a type-K order-preserving (sub)homogeneous map, and vice versa. In the literature, an order-preserving plus-(sub)homogeneous map is usually referred to as a *(sub)topical* map. Several interesting examples of topical maps arise in optimal control and scheduling theory (max-plus maps), in Markov decision theory and stochastic game, (see Section 1.5 in [134] for an introduction and references).

[134] Lemmens and Nussbaum (2012), *Nonlinear Perron-Frobenius Theory*.

3.4 Non-expansiveness of maps

Order-preserving maps possessing one of the variants of the homogeneity property discussed in Section 3.3 are non-expansive under some metric.

Definition 3.4.1 *A map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be,* non-expansive, *with respect to a metric $d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}_+$, if $\forall x, y \in \mathbb{R}^n$ and*

$\alpha \in \mathbb{R}$ it holds

$$d(f(x), f(y)) \leq d(x, y).$$

As we shall see later, the non-expansiveness property is a powerful tool in the study of the iterative behavior of order-preserving maps. It lies at the heart of many arguments in nonlinear Perron–Frobenius theory. The purpose of this chapter is to discuss the relation between order-preserving maps and non-expansive maps. In particular, three main facts are discussed:

1. Order-preserving and subhomogeneous maps on $\mathrm{int}(\mathbb{R}^n_+)$ are non-expansive under the Tomphson's metric d_T;
2. Order-preserving and plus-subhomogeneous maps on $\mathbb{R}^n$ are non-expansive under the sup-metric d_∞;
3. The metric spaces $(\mathrm{int}(\mathbb{R}^n_+), d_T)$ and $(\mathbb{R}^n, d_\infty)$ can be isometrically embedded into each other.

Next, the Thompson's metric and the sup-metric are formally defined.

Definition 3.4.2 *The* Thompson's metric d_T *and the* sup-metric d_∞ *induced by the sup-norm are defined by*

$$d_T(x, y) = \begin{cases} \ln\left(\max\left\{\max_i \frac{x_i}{y_i}, \max_i \frac{y_i}{x_i}\right\}\right) & \text{if } x \sim y \\ 0 & \text{if } x = y = \mathbb{0} \\ \infty & \text{otherwise} \end{cases}$$

$$d_\infty(x, y) = \|x - y\|_\infty = \max_i |x_i - y_i|.$$

The next result is taken from [4] but it is stated here for the standard positive cone $K = \mathbb{R}^n_+$.

[4] Akian et al. (2006), 'Iteration of order preserving subhomogeneous maps on a cone'.

Proposition 3.4.1 *[4] An order-preserving map $f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is non-expansive, w.r.t. the Tomphson's metric d_T, if and only if is subhomogeneous.*

For the sake of readability, some times the short expression *Thompson non-expansive* will be used instead of *non-expansive w.r.t. the Thompson's metric*. The next result is taken from [47] but it is stated here for sup-metric.

[47] Crandall and Tartar (1980), 'Some relations between nonexpansive and order preserving mappings'.

Proposition 3.4.2 *[47] An order-preserving map $f : \mathbb{R}^n \to \mathbb{R}^n$ is non-expansive, w.r.t. the sup-metric d_∞, if is plus-subhomogeneous;*

For the sake of readability, some times the short expression *sup-norm non-expansive* will be used instead of *non-expansive w.r.t. the sup-metric*. The results of Proposition 3.4.1 and Proposition 3.4.2 can also be obtained one from the other by using the log-exp transform (3.4), as the following proposition suggests..

Proposition 3.4.3 *[134]*

- *The map L is an isometry from $(int(\mathbb{R}^n_+), d_T)$ onto $(\mathbb{R}^n, d_\infty)$.*
- *The map E is an isometry from $(\mathbb{R}^n, d_\infty)$ onto $(int(\mathbb{R}^n_+), d_T)$.*

[134] Lemmens and Nussbaum (2012), *Nonlinear Perron-Frobenius Theory*.

This allows us to translate results from the multiplicative homogeneous setting in Definition 3.3.2 to the additively homogeneous setting in Definition 3.3.1 3.3.1 and vice versa. This strategy is used in the next section to prove our main results.

3.5 Stability analysis via nonlinear PF theory

It is easy to realize that the sequence of points generated by the recursive iteration of a map $f(\cdot)$ starting from a point $x_0 \in \mathbb{R}$ corresponds to the *trajectory* of a dynamical systems $x(k+1) = f(x(k))$ with initial condition $x(0) = x_0$.

Basic notion about trajectories and periodic points are given in Appendix A.1.

Definition 3.5.1 *The* trajectory $\mathcal{T}(x_0, f)$ *of system* (3.1) *with initial state $x_0 \in \mathbb{R}$ is given by*

$$\mathcal{T}(x, f) = \left\{ f^k(x_0) : k \in \mathbb{Z} \right\}.$$

If the map f is clear from the context, we simply write $\mathcal{T}(x_0)$.

Definition 3.5.2 *A trajectory $\mathcal{T}(x_0, f)$ is said to be:*

- bounded *if it has both upper and lower bounds, i.e., there exist $a, b \in \mathbb{R}$ such that $x \in [a, b]$ for all $x \in \mathcal{T}(x_0, f)$; otherwise, it is said to be* unbounded*;*
- periodic *if it is bounded and if there exists $p \in \mathbb{Z}_+$ such that $f^p(x) = x$ for all $x \in \mathcal{T}(x_0, f)$;*

Note that all points in a periodic trajectory are periodic points of period $p \in \mathbb{Z}_+$, i.e., $f^p(x) = x$; a periodic point of period $p = 1$ is said to be a *fixed* of the map f or an *equilibrium* point of the system, for which holds $f(x) = x$. Let us consider the following map $f : \mathbb{R}^3 \to \mathbb{R}^3$ as an example

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_1 \\ \beta x_2 \\ x_3 \end{bmatrix}, \qquad \alpha, \beta \in \mathbb{R}_+$$

and consider a trajectory $\mathcal{T}(x, f)$, given next

$$\mathcal{T}(\bar{x}, f) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} \alpha x_2 \\ \beta x_3 \\ x_1 \end{bmatrix}, \begin{bmatrix} \alpha\beta x_3 \\ \beta x_1 \\ \alpha x_2 \end{bmatrix}, \begin{bmatrix} \alpha\beta x_1 \\ \alpha\beta x_2 \\ \alpha\beta x_3 \end{bmatrix}, \begin{bmatrix} \alpha^2\beta x_2 \\ \alpha\beta^2 x_3 \\ \alpha\beta x_1 \end{bmatrix}, \cdots \right\}.$$

We consider the following cases:

- If $\alpha\beta = 1$ and $\alpha \neq \beta \neq 1$, then
  - All points $\bar{x} = \gamma[2, 1, 2]$ with $\gamma \in \mathbb{R}_+$ are fixed points, thus the trajectory contains only one point $\mathcal{T}(\bar{x}, f) = \{\bar{x}\}$;
  - All other points $x \neq \bar{x}$ are periodic points of period $p = 3$, thus the trajectory contains only 3 points $\mathcal{T}(\bar{x}, f) = \{[x_1, x_2, x_3]^\intercal, [\alpha x_2, \beta x_3, x_1], [x_3, \beta x_2, \alpha x_2]\}$.
- If $\alpha = \beta = 1$, then all points $x \in \mathbb{R}^3$ are fixed points, thus all trajectories contains only one point $\mathcal{T}(x, f) = \{x\}$.
- If $\alpha\beta < 1$, then all trajectories are bounded and converge to the point $\mathbb{0} = [0, 0, 0]^\intercal$, which is the only fixed point of map f, i.e., $f(\mathbb{0}) = \mathbb{0}$ and $\mathcal{T}(\mathbb{0}, f) = \{\mathbb{0}\}$.
- If $\alpha\beta > 1$, then all trajectories are unbounded but $\mathcal{T}(\mathbb{0}, f)$.

Thus, studying the properties of the iterative behavior of a map amounts to the analysis of the trajectories of the associated dynamical systems. Here, we consider dynamical systems which are ruled by positive, type-K order-preserving maps possessing one of the variation of homogeneity property, object of the nonlinear Perron-Frobenius theory introduced in the previous sections.

For a nonlinear positive map $f(\cdot)$, the trajectories of the system $x(k+1) = f(x(k))$ starting at different points show in general a very different convergence behavior. For example, one trajectory may tend to infinity whereas another one tends to zero or still another one converges to a point in the interior of the underlying cone. This is true even in one dimension as exemplified by the system $x(k+1) = x^2(k)$. The situation drastically changes if the positive system is linear: a linear system defined by a primitive matrix shows a uniform behavior for all starting points in $\mathbb{R}^n_+ \setminus \{0\}$ that is either

- all trajectories tend to infinity,
- all trajectories tend to zero,
- all trajectories tend to converge to a fixed point in the interior of the standard positive cone.

This property is called limit set trichotomy. For a system given by a primitive matrix this property follows from classical Perron–Frobenius Theory (see Theorem 3.1.2), where the trichotomy is due to the three cases whether the dominant eigenvalue is greater or smaller or equal to 1, respectively. The interested reader is referred to [125] for a comprehensive coverage of positive maps, with particular attention to Chapter 6 and several generalization of this limit set trichotomy to more general positive maps. In Figure 3.6 a stylized picture of limit set trichotomy illustrates this in one dimension. Respectively, map f_1, f_2, f_3 stand for the case that for all points in the interior of the standard positive cone the trajectory tends to zero, to a unique fixed point, and to infinity. An example of these functions is $f_1(x) = \text{atan}(x)$, $f_2(x) = \sqrt{x}$, $f_3 = x + \ln(x+1)$. Clearly, in n dimension the analysis becomes even more complicated due to the possible presence of periodic points.

[125] Krause (2015), *Positive dynamical systems in discrete time: theory, models, and applications*.

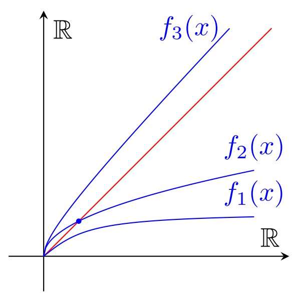

Figure 3.6: Limit set trichotomy in one dimension.

This motivates the study of the iterative behavior of sup-norm non-expansive maps which have some striking properties: their bounded trajectories always converge to a periodic trajectory and they are either all bounded or unbounded. These two facts are formally stated in the next propositions [134].

[134] Lemmens and Nussbaum (2012), *Nonlinear Perron-Frobenius Theory*.

Proposition 3.5.1 *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a sup-norm non-expansive map and the trajectory $\mathcal{T}(x, f)$ of system (3.1) starting at $x \in \mathbb{R}^n$ is bounded, then it converges to a periodic trajectory, i.e., there exists an integer $p \geq 1$ and a periodic point $\bar{x} \in \mathbb{R}^n$ with period p such that*

$$\lim_{k\to\infty} f^{kp}(x) = \bar{x}.$$

Proposition 3.5.2 *If $f : \mathbb{R}^n \to \mathbb{R}^n$ be a sup-norm non-expansive map, then trajectories are all either bounded or unbounded.*

If the map is merely nonexpansive, then the sequence generated by their recursive iteration may fail to converge to a fixed point. For instance, this is the case for $f(x) = -x$. Thus, our goal is the identification of classes of maps possessing the non-expansiveness property such that all trajectories are bounded and no trajectory tends to a periodic point, i.e., all trajectories converge to a fixed point of the map or equivalently the system converges to one of its equilibrium points. For linear maps, it has been proved that a necessary and sufficient condition ensuring the convergence to a fixed point is the *averagedness* of the map [**Belgioioso18**, 41], while for nonlinear maps the averagedness property has been shown to be a sufficient condition since a long time [23], but not necessary.

[**Belgioioso18**] **Belgioioso18** (**Belgioioso18**), **Belgioioso18**.

[41] (2020), 'Convergence in uncertain linear systems'.

[23] Bauschke, Combettes, et al. (2011), *Convex analysis and monotone operator theory in Hilbert spaces*.

By taking inspiration from nonlinear Perron-Frobenius theory, here the focus is on maps preserving the order induced by the standard positive cone and that possess also one of the variants of the homogeneity property, thus ensuring their non-expansiveness with respect to some norm. In particular, the novel version of order-preservation originally presented in [112], the so-called *type-K order-preservation*, is the key ingredient in the proofs of our results, in particular convergence results are carried out for two classes of maps:

[112] Jiang (1996), 'Sublinear discrete-time order-preserving dynamical systems'.

- Type-K order-preserving and subhomogeneous maps in the standard positive cone $\mathbb{R}^n_+$, see Theorem 3.5.3;
- Type-K order-preserving and plus-subhomogeneous maps in the whole real vector space $\mathbb{R}^n$, see Theorem 3.5.4.

It is interesting to notice that the considered classes of maps are not necessary averaged in the sense of [23], thus enlarging the set of nonlinear maps for which there exists global convergence results.

For positive maps which are also order-preserving and subhomogeneous, existing results do not provide any condition to ensure convergence to a fixed point, but only to periodic points [133]. Furthermore, the analysis is carried out only in the interior of the standard positive cone, i.e., $x \in \text{int}(\mathbb{R}^n_+)$, and, to the best of our knowledge, no result provides any information about trajectories whose initial state lies in the boundary of $\mathbb{R}^n_+$. Our aim is thus to fill this void by considering the stricter order-preservation property, called *type-K order-preservation*, for which convergence to a fixed point from any initial state $x \in \mathbb{R}^n_+$ and not only for $x \in \text{int}(\mathbb{R}^n_+)$ is proved. This result is given in next theorem.

[133] Lemmens (2006), 'Nonlinear Perron-Frobenius theory and dynamics of cone maps'.

Theorem 3.5.3 *Let a map $f : \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n_+$ be type-K order-preserving and subhomogeneous. If f has at least one positive fixed point, then it holds that*

$$\lim_{k \to \infty} f^k(x) = \bar{x}, \quad \forall x \in \mathbb{R}^n_+$$

where $\bar{x} \in F(f)$ is a fixed point of $f(\cdot)$.

By exploiting the isometry in Proposition 3.4.3, the convergence to the set of fixed points for maps being type-K order-preserving and plus-subhomogeneus acting in the whole real vector space $\mathbb{R}^n$ is proved.

Theorem 3.5.4 *Let a map $f : \mathbb{R}^n \to \mathbb{R}^n$ be type-K order-preserving and plus-subhomogeneous. If f has at least one fixed point, then it holds that*

$$\lim_{k\to\infty} f^k(x) = \bar{x}, \quad \forall x \in \mathbb{R}^n_+$$

where $\bar{x} \in F(f)$ is a fixed point of $f(\cdot)$.

Proof of Theorem 3.5.3

Before giving the proof of this theorem it is necessary to introduce some useful lemmas.

Lemma 3.5.5 *Let a map $f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be type-K order-preserving. For all $x \in \mathbb{R}^n_+$ it holds that $f_i^k(x) > 0$ for all i such that $x_i > 0$ and $k \geq 1$.*

Proof. For any $x \in \mathbb{R}^n_+$ let $I(x) \subset \{1, \ldots, n\}$ be such that $x_i > 0$ for $i \in I(x)$ and $x_i = 0$ otherwise. Since $\mathbb{0} \leq x$, by type-K order-preservation of f follows $f(\mathbb{0}) \leq f(x)$. More precisely it holds $f_i(x) > f_i(\mathbb{0}) \geq 0$ for $i \in I(x)$ and $f_i(x) \geq f_i(\mathbb{0}) \geq 0$ otherwise, implying $I(x) \subseteq I(f(x))$. By induction, $I(x) \subseteq I(f^k(x))$, i.e., $f_i^k(x) > 0$ for all $i \in I(x)$, completing the proof. □

Lemma 3.5.6 *Let a map $f : \mathbb{R}^n_+ \to \mathbb{R}^n{}_+$ be type-K order-preserving and subhomogeneous. For all $x \in \mathbb{R}^n_+$ there exists a part $P \in \mathcal{P}(\mathbb{R}^n_+)$ and an integer $k_0 \in \mathbb{Z}$ such that $f^k(x) \in P$ for all $k \geq k_0$.*

Proof. Since f is order-preserving and subhomogeneous, then f is non-expansive under Thompson's metric (see Definition 3.4.2) by Proposition 3.4.1. Thus, $x \sim y$ implies $f(x) \sim f(y)$. This can be easily proved by noticing that $d_T(f(x), f(y)) \leq d_T(x, y) < \infty$ since $x \sim y$. This means that f maps parts into parts, i.e., for all $x \in \mathbb{R}^n_+$ and $x' \in [x] = P_{I_0}$ it holds $f(x') \in [f(x)] = P_{I_1}$. By Lemma 3.5.5 it follows $P_{I_0} \leq P_{I_1}$ and therefore $[x] \leq [f(x)]$. Generalizing, one can state that $f^k(x) \in P_{I_k}$ with $k \in \mathbb{Z}$ and $I_k \subseteq I_{k+1} \subseteq \{1, \ldots, n\}$. There exists $k_0 \in \mathbb{Z}$ such that $I_k = I_{k_0}$ for all $k > k_0$ and thus $P_k = P_{k_0}$. This completes the proof. □

Lemma 3.5.7 *Let a map $f : \mathbb{R}^n_+ \to \mathbb{R}^n{}_+$ be type-K order-preserving and subhomogeneous. If f has a positive fixed point $\bar{x} \in \mathbb{R}^n_+$, then for* *all $x \in \mathbb{R}^n_+$ the trajectory $\mathcal{T}(x)$ is bounded.*

Proof. By Proposition 3.4.3, function $g = \log \circ f \circ \exp$ is a sup-norm non-expansive map that has the same dynamical properties as f for all $x \in \mathbb{R}^n_+$. By Proposition 3.5.2, one of the two cases can occur:

(i) all trajectories $\mathcal{T}(\log(x), g)$ are unbounded;
(ii) all trajectories $\mathcal{T}(\log(x), g)$ are bounded.

Since f has a fixed point $x_f \in \mathbb{R}^n_+$, such that $f(x_f) = x_f$, then $x_g = \log(x_f)$ is a fixed point of g, i.e., $g(x_g) = x_g$. The trajectory $\mathcal{T}(\log(x_f), g)$ is obviously bounded and therefore case (ii) holds.

By Lemma 3.5.6, $\mathbb{R}^n_+$ is partitioned into two disjoint sets S_1, S_2 such that if for x there exists $k_0 \in \mathbb{Z}$ such that $f^{k_0}(x) \in \mathbb{R}^n_+$, then $x \in S_1$, otherwise $x \in S_2$. Next, the two cases are analyzed.

1) For all $x \in S_1$, by Lemma 3.5.6, it holds that $f^k(x) \in \mathbb{R}^n_+$ for all $k \geq k_0$. Let $x_0 = f^{k_0}(x)$. Since case (ii) holds $\mathcal{T}(\log(x_0), g)$ is bounded, because of the isometry also $\mathcal{T}(x_0, f)$ is bounded, and therefore also $\mathcal{T}(x, f)$. Therefore, for all $x \in S_1$ trajectories $\mathcal{T}(x, f)$ are bounded.

2) For all $x \in S_2$, by Lemma 3.5.6, there exists $k_0 \in \mathbb{Z}$ such that $f^k(x) \in P_I$ with $I(x) \subset N = \{1, \ldots, n\}$ for all $k \geq k_0$. Without loss of generality, here it is assumed that $I = \{1, \ldots, m\}$, where $m < n$. Let $x = [z_1^\top, z_2^\top]^\top$ with $z_1 \in \mathbb{R}^m_+$ and consider the following m-dimensional map $f^* : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ defined by

$$f_i^*(z_1) = f_i(z_1, z_2), \quad z_2 = \mathbb{0},$$

with $i \in I(x)$. It is not difficult to check that f^* is still subhomogeneous and type-K order-preserving. Accordingly, $g^* = \log \circ f^* \circ \exp$ is a sup-norm non-expansive map that has the same dynamical properties as f^* for all $x \in \mathbb{R}^m_+$. The main point now is to prove that if (ii) occurs then all trajectories $\mathcal{T}(\log(z_1), g^*)$ are also bounded. To this aim, first is is shown that for all $i \in I(x)$ it holds

$$g_i^*(z_1) \leq g_i(z_1, z_2). \tag{3.5}$$

Since both the exponential and the logarithmic functions are strictly increasing, (3.5) is equivalent to

$$f_i^*(z_1) \leq f_i(z_1, z_2). \tag{3.6}$$

By definition, (3.6) holds if $z_2 = 0$. If $z_2 \neq 0$, for any $x = [z_1^\top, z_2^\top]^\top$ consider $\bar{x} = [z_1^\top, \bar{z}_2^\top]^\top$ such that $\bar{z}_2 = \mathbb{0}$. Since f is order-preserving, for all $i \in I$ it holds that $f_i(\bar{x}) \leq f_i(x)$, which is equivalent to write $f_i(z_1, \bar{z}_2) \leq f_i(z_1, z_2)$. By definition, $f_i^*(z_1) = f_i(z_1, \bar{z}_2)$. Therefore, $f_i^*(z_1) \leq f_i(z_1, z_2)$ for all $z_2 \neq 0$, i.e., (3.6) and (3.5) hold. Suppose that (ii) occurs and there exist $\hat{z}_1 \in \mathbb{R}_+^m$ such that $\mathcal{T}(\log(\hat{z}_1), g^*)$ is unbounded. By (3.5) it is clear that given $\hat{x} = [\hat{z}_1^\mathsf{T}, z_2^\mathsf{T}]^\mathsf{T}$ the trajectory $\mathcal{T}(\log(\hat{x}), g)$ is also unbounded, contradicting (ii). Let $x_0 = f^{k_0}(x)$. Since all trajectories $\mathcal{T}(\log(x_0), g)$ are bounded, because of the isometry also $\mathcal{T}(x_0, f)$ is bounded, and therefore also $\mathcal{T}(x, f)$. Therefore, for all $x \in S_2$ trajectories $\mathcal{T}(x, f)$ are bounded. □

Finally,using the above lemmas, the proof of Theorem 3.5.3 is carried out. By Lemma 3.5.7, trajectories of positive, subhomogeneous and type-K order-preserving maps with a positive fixed point are bounded for all $x \in \mathbb{R}_+^n$. By Proposition 3.5.1, each bounded trajectory converges to a periodic point and, by Theorem 2.3 in [112], all periodic points are fixed points. Therefore,

[112] Jiang (1996), 'Sublinear discrete-time order-preserving dynamical systems'.

$$\lim_{k\to\infty} f^k(x) = \bar{x}, \quad \forall x \in \mathbb{R}_+^n$$

where $\bar{x} \in F(f)$ is a fixed point of $f(\cdot)$, completing the proof.

Proof of Theorem 3.5.4

If map $f : \mathbb{R}^n \to \mathbb{R}^n$ is type-K order-preserving, then the log-exp transform $g : \mathrm{int}(\mathbb{R}^n) \to \mathrm{int}(\mathbb{R}^n)$ of $f(\cdot)$ given by $h = \exp \circ f \circ \log$ for $z \in \mathrm{int}(\mathbb{R}^n)$ is still type-K order-preserving. This can be easily proved by noticing that the exponential and logarithmic functions are strictly increasing. Furthermore, plus-subhomogeneity of $f(\cdot)$ is inherited as subhomogeneity by $g(\cdot)$, as shown next

$$\begin{aligned} g(\alpha x) = e^{f(\ln(\alpha x))} &= e^{f(\ln(\alpha)+\ln(x))} \geq e^{f(\ln(\alpha))+f(\ln(x))} \\ &\geq e^{\ln(\alpha)+f(\ln(x))} \geq e^{\ln(\alpha)} e^{f(\ln(x))} \geq \alpha g(x) \end{aligned}$$

Let $\bar{x} \in \mathbb{R}^n$ be a fixed point such that $f(\bar{x}) = \bar{x}$. Its mapping $\bar{z} = E(\bar{x}) = e^{\bar{x}}$ is a fixed point of $h(\cdot)$, i.e., $g(\bar{z}) = \bar{z}$.

Now, Theorem 3.5.3 can be exploited by noticing that map $g(\cdot)$ is type-K order preserving and subhomogeneous and has a positive fixed point $\bar{z} \in \mathrm{int}(\mathbb{R}_+^n)$, thus concluding that $g(\cdot)$ does not have any periodic point with period $p > 1$, i.e., all its periodic points are fixed points. Since eventually periodic points of $f(\cdot)$ would be mapped into periodic points of $g(\cdot)$ by the exponential function, and vice versa by the logarithmic function, therefore $f(\cdot)$ does not have periodic points. By Proposition 3.5.2, map $f(\cdot)$ is non-expansive under the sup-norm and then by Proposition 3.5.2, only one of the following two cases can occur:

(i) all trajectories $\mathcal{T}(x, f)$ are unbounded;
(ii) all trajectories $\mathcal{T}(x, f)$ are bounded.

The presence of a fixed point $\bar{x} \in \mathbb{R}$ implies that the trajectory $\mathcal{T}(\bar{x}, f)$ is bounded and so all others, as in case (ii). Since all trajectories are bounded but no periodic trajectories exist, therefore all trajectories converge to a fixed point, thus completing the proof.

4 Stability and consensus of discrete-time MASs

In this chapter MASs in which the agents are single discrete-time integrators are considered. The dynamics of each agent is described by

$$x_i(k+1) = f_i(x_i(k), x_j(k) : j \in \mathcal{N}_i). \tag{4.1}$$

where $x_i(k) \in \mathbb{R}$ represents the state of the i-th agent at time k. Denoting $x = [x_1, \cdots, x_n]^\intercal$ the state of the MAS, with $n \in \mathbb{N}$ being the number of the agents, the global dynamics is written as

$$x(k+1) = f(x(k)). \tag{4.2}$$

In the linear case classical Perron-Frobenius theory is crucial in the convergence analysis of MASs, as recalled in Chapter 3. A topic that captured the attention of many researchers is the consensus problem [176], where the objective is to design local interaction rules among agents such that their state variables converge to the same value, the so called agreement or consensus state. Indeed, in one of the seminal works on this topic [110], the authors established criteria for convergence to a consensus state for MASs whose global dynamics can be represented by linear time-varying systems with nonnegative row-stochastic state transition matrices. The most notable aspect of this approach was a novel setting based on Perron-Frobenius theory for proving convergence based on algebraic theory and graph theory instead of Lyapunov theory. This allows one to study systems for which finding a common Lyapunov function to establish convergence is difficult or even impossible: such is the case of switched linear systems for which does not exist a common quadratic Lyapunov function, as shown in [177].

[176] Olfati-Saber et al. (2007), 'Consensus and cooperation in networked multi-agent systems'.

[110] Jadbabaie et al. (2003), 'Coordination of groups of mobile autonomous agents using nearest neighbor rules'.

[177] Olshevsky and Tsitsiklis (2008), 'On the Nonexistence of Quadratic Lyapunov Functions for Consensus Algorithms'.

Along this line of thought, in this chapter the results given in Section 3.5 which are based on nonlinear Perron-Frobenius theory [134] are exploited to study stability of discrete-time MASs without Lyapunov based arguments. MASs evolving in a real vector state space $\mathbb{X} \subseteq \mathbb{R}^n$ with nonlinear dynamics given by map $f : \mathbb{X} \to \mathbb{X}$ are considered. In particular, two classes of dynamics are considered:

[134] Lemmens and Nussbaum (2012), *Nonlinear Perron-Frobenius Theory*.

- Type-K order-preserving and subhomogeneous maps in the standard positive cone $\mathbb{R}^n_+$;

- Type-K order-preserving and plus-subhomogeneous maps in the whole real vector space $\mathbb{R}^n$.

In Section 3.5 the convergence properties of the iterative behavior of these classes of maps was analyzed by exploiting fixed-point theory and nonlinear Perron-Frobenius theory instead of Lyapunov theory. The main contribution of this chapter is the application of these new mathematical instruments to the analysis of the stability and convergence to consensus of MASs. More precisely, for each of these classes:

- Sufficient conditions on the local interaction rule of a generic agent are proposed (such a rule can be potentially different for each agent, i.e., heterogeneous) which guarantees that the system falls into the considered class and is stable if a positive equilibrium point exists;
- Sufficient conditions linking the topology of the network and the structure of the local interaction rules are proposed, which guarantees the achievement of a consensus state, i.e., the network state in which all state variables have the same value.

4.1 Type-K order-preserving and subhomogeneous systems

The dynamical systems of interest possess the property of order-preservation with respect to the standard positive cone and subhomogeneous, which are object of nonlinear Perron-Frobenius theory [134]. It is worth mentioning that there also exists a *concave Perron-Frobenius theory* [125], which deals with *concave* maps. For maps acting on a cone, concavity implies both order-preservation and subhomogeneity, but the vice versa does not hold. Thus, the class of order-preserving and subhomogeneous maps considered in this chapter is more general. In the next subsections, the related literature is analyzed.

[134] Lemmens and Nussbaum (2012), *Nonlinear Perron-Frobenius Theory*.

[125] Krause (2015), *Positive dynamical systems in discrete time: theory, models, and applications*.

About homogeneous systems

Most of the current literature is mainly concerned with systems evolving in the interior of the standard positive cone ruled by maps which are order-preserving and homogeneous: these maps need not be concave and vice versa. These maps characterize a rich class of dynamical systems, for which there is extensive literature (see [95, 172] for references). Although this class of systems has been widely studied in the literature in continuous-time, there are few results which consider the discrete–time case. The interested reader is referred to Chapter 5 for the continuous-time counterpart of our results and the corresponding literature review. A fundamental result was presented by Gaubert and Gunawardena [91] generalizing a result of Nussbaum [173]. These authors prove that an eigenvector in the interior of the standard positive cone is guaranteed to exist under the assumption that a graph $\mathcal{G}(f)$ associated to the map f is strongly connected. Furthermore, the associated eigenvalue is unique and thus it may be considered as a spectral radius, in a limited sense. It is easy to imagine that if the spectral radius is greater than (resp., smaller than, equal to) 1, then the recursive iteration f^k of the map diverges (resp., converge to zero or converges to a fixed point). Despite this strong result, the iteration of order-preserving and homogeneous maps does not always converge to a fixed point of the map, since no a priori bound on the spectral radius is given and also because of the presence of periodic trajectories can not be avoided a priori.

[95] Gunawardena (2003), 'From max-plus algebra to nonexpansive mappings: a nonlinear theory for discrete event systems'.

[172] Nussbaum (1988), *Hilbert's Projective Metric and Iterated Nonlinear Maps*.

[91] Gaubert and Gunawardena (2004), 'The Perron-Frobenius theorem for homogeneous, monotone functions'.

[173] Nussbaum (1986), 'Convexity and log convexity for the spectral radius'.

Next, recent results related to order-preserving and homogeneous dynamical systems are reviewed. To the best of our knowledge, no application to multi-agent systems has been discussed. Order-preserving and homogeneous systems are a natural extension of positive linear systems, for which there is a well-developed theory rooted in the Perron-Frobenius theory of nonnegative matrices [69]. Therefore, much effort has done to extend the properties of positive linear systems. One of the main results is the global asymptotic stability of order-preserving and homogeneous systems [49, 50, 99, 131] with extensions to the case of homogeneity of order greater than one one [64] and also other newly defined kind of homogeneity [199, 200]. Attention has also been paid to the robustness of homogeneous maps under time delays. In particular, robustness results are provided for systems with constant [72] and time-varying delays [65]. Finally, another interesting object of study are order-preserving switched homogeneous systems, a superclass of switched linear systems, which have been the subject of interest since the early work of [75] and more recently [19, 107, 248]. Similar results may be possibly established for the class of subhomogeneous maps, which includes homogeneous maps.

[69] Farina and Rinaldi (2011), *Positive linear systems: theory and applications*.

[99] Hammouri and Benamor (1999), 'Global stabilization of discrete-time homogeneous systems'.

[49] Dashkovskiy et al. (2006), 'Discrete time monotone systems: Criteria for global asymptotic stability and applications'.

[50] Dashkovskiy et al. (2007), 'An ISS small gain theorem for general networks'.

[131] Lazar et al. (2013), 'On stability analysis of discrete-time homogeneous dynamics'.

[64] Doban and Lazar (2014), 'Stability analysis of discrete–time general homogeneous systems'.

[199] Sanchez et al. (2017), 'A homogeneity property of a class of discrete-time systems'.

[200] Sanchez et al. (2019), 'A homogeneity property of discrete-time systems: Stability and convergence rates'.

About Moreau's convexity condition

Most approaches that aim to establish convergence to consensus for some class of nonlinear MAS fall in the general convexity theory of [160], i.e., each agent's next state is strictly inside the convex hull spanned by its own state and the state value of its neighbors. The class of systems studied in this chapter is not limited to Moreau's theory, as our application in Section 4.4 shows. In fact, the considered dynamical system is a distributed algorithm to estimate the maximum among all initial agents' state, therefore not satisfying a strict convex condition. The study of dynamical equations not satisfying a strict convexity assumption is beyond the scope of [160].

[72] Feyzmahdavian et al. (2013), 'Exponential stability of homogeneous positive systems of degree one with time-varying delays'.
[65] Dong (2015), 'On the decay rates of homogeneous positive systems of any degree with time-varying delays'.
[75] Filippov (1980), 'Stability conditions in homogeneous systems with arbitrary regime switching'.
[107] Holcman and Margaliot (2002), 'Stability analysis of second-order switched homogeneous systems'.
[248] Zhang et al. (2007), 'On stability of switched homogeneous nonlinear systems'.
[19] Athanasopoulos and Lazar (2014), 'Alternative stability conditions for switched discrete time linear systems'.
[160] Moreau (2005), 'Stability of multiagent systems with time-dependent communication links'.
[77] Forni and Sepulchre (2016), 'Differentially positive systems'.

About differential positivity

The differential positivity framework, developed by Forni et al. in [77], addresses a problem setup similar to the one in this chapter. The main common point between the results of this chapter and the work in [77] is the fact that both consider *positive* systems. Positivity in [77] is intended in the sense of cone invariance; positivity is said to be *strict* if the boundary of the cone is eventually mapped to the interior of the cone. In contrast, results in this chapter are restricted to systems with a state space $\mathcal{X} = \mathbb{R}$ and a constant invariant cone $K = \mathbb{R}^n_+$. For such systems, by Theorem 1 in [77], it follows

$$\begin{aligned} \text{differential positivity} &\Leftrightarrow \text{order-preservation}, \\ \text{strict differential positivity} &\Leftrightarrow \text{strong order-preservation}. \end{aligned}$$

The results in [77] are limited to strictly differentially positive systems: no convergence results are provided for differentially positive systems. A simple example given next shows that the class of maps addressed in our work is differentially positive but not strict,

$$x(k+1) = Ax(k), \quad A = \begin{bmatrix} 1 & 0 \\ \gamma & 1-\gamma \end{bmatrix}, \quad \gamma \in (0,1).$$

Map Ax is not strictly differentially positive because it does not map the boundary of the positive orthant to its interior and this is true for any $x = [0, \alpha]^\intercal$, with $\alpha \in \mathbb{R}$. However, map A is type-K order-preserving since it is Metzler with strictly positive diagonal. Exploiting the notion of type-K order-preservation in the context of differential positivity may enlarge the class of systems that can be studied under a general framework encompassing the two approaches.

4.2 Stability and consensus analysis

Before establishing the main results of this section, we formally define the graph associated to a MAS when the local interaction rules are assumed to be nonlinear and differentiable.

Definition 4.2.1 *Given a differentiable map* $f : X \to X$ *with* $X \subseteq \mathbb{R}^n$*, its inference graph* $\mathcal{G}(f) = (V, E)$ *has a set of* n *nodes* $V = \{1, \ldots, n\}$ *and a set of directed edges* $E \subseteq V \times V$*. An edge* $(i, j) \in E$ *from node* i *to node* j *exists if*

$$\frac{\partial}{\partial x_j} f_i(x) \neq 0, \qquad \forall x \in X \setminus S,$$

where S *is a set of measure zero in* X*.*

Stability analysis

Next theorem gives sufficient conditions on the structure of the local interaction rules of a discrete-time MAS as in (4.1) so that the global map is a positive, type-K order-preserving and subhomogeneous map. Such a dynamical system belongs to the class of systems considered in Theorem 3.5.3, and thus the convergence to an equilibrium point of the system is ensured for any initial condition in $\mathbb{R}^n_+$.

Theorem 4.2.1 *Consider* n *agents evolving according to*

$$x_i(k+1) = f_i(x_i(k), x_j(k) : j \in \mathcal{N}_i).$$

If the MAS has at least one positive equilibrium point and if the set of differentiable positive local interaction rules $f_i : \mathbb{R}^n_+ \to \mathbb{R}_+$*, with* $i = 1, \ldots, n$*, satisfies the two conditions:*

(*i*) $\partial f_i / \partial x_i > 0$ *and* $\partial f_i / \partial x_j \geq 0$ *for* $i \neq j$*;*
(*ii*) $\alpha f_i(x) \leq f_i(\alpha x)$ *for any* $\alpha \in [0, 1]$ *and* $x \in \mathbb{R}^n_+$*;*

then the MAS asymptotically converges to one of its equilibrium points for any positive initial state $x(0) \in \mathbb{R}^n_+$*.*

Proof. The proof starts by establishing an equivalence between the properties $(i) - (ii)$ in the statement of the theorem and properties (a)-(b) shown next:

(a) f is type-K order-preserving;
(b) f is subhomogeneous;

Each equivalence is proved next:

- $[(i) \Leftrightarrow (a)]$ due to Kamke-like condition in Proposition 3.2.1.
- $[(ii) \Leftrightarrow (b)]$ by Definition 3.3.1 of a subhomogeneous map, subhomogeneity can be verified element-wise for map f, thus the equivalence follows.

If conditions (i)-(ii) hold for all local interaction rules f_i with $i = 1, \ldots, n$, since by assumption map f has at least one positive fixed point, the result in Theorem 3.5.3 ensures that for all positive initial conditions, the state trajectories of the MAS converge to one of its positive equilibrium points. □

In the next theorem, sufficient conditions on the structure of the local interaction rules of a discrete-time MAS as in (4.1) so that the global map is a positive, type-K order-preserving and plus-homogeneous map, thus falling within the class of systems considered in Theorem 3.5.4 and ensuring the stability of the MAS for any initial condition in $\mathbb{R}^n$.

Theorem 4.2.2 *Consider n agents evolving according to*

$$x_i(k+1) = f_i(x_i(k), x_j(k) : j \in \mathcal{N}_i).$$

If the MAS has at least one positive equilibrium point and if the set of differentiable local interaction rules $f_i : \mathbb{R}^n \to \mathbb{R}$, with $i = 1, \ldots, n$, satisfies the next conditions:

(i) $\partial f_i / \partial x_i > 0$ *and* $\partial f_i / \partial x_j \geq 0$ *for* $i \neq j$;
(ii) $f_i(x + \alpha \mathbb{1}) = f_i(x) + \alpha \mathbb{1}$ *for any* $\alpha \in \mathbb{R}$;

then the MAS asymptotically converges to one of its equilibrium points for any initial state $x(0) \in \mathbb{R}^n$.

Proof. The proof starts by establishing equivalence relationships between the properties $(i) - (ii)$ of the local interaction rules of the MAS listed in the statement of the theorem and properties (a)-(b) shown next:

(a) f is type-K order-preserving;

(b) f is plus-homogeneous;

Each equivalence reads as:

- $[(i) \Leftrightarrow (a)]$ due to Kamke-like condition in Proposition 3.2.1.
- $[(ii) \Leftrightarrow (b)]$ by Definition 3.3.1 of a plus-homogeneous map, plus-homogeneity can be verified element-wise for map f, thus the equivalence follows.

If conditions (i)-(ii) hold for all local interaction rules f_i with $i = 1, \dots, n$, since by assumption map f has at least one positive fixed point, the result in Theorem 3.5.4 can be exploited to establish that for all positive initial conditions, the state trajectories of the MAS converge to one of its positive equilibrium points. □

Consensus analysis

As a special case, the consensus problem for the two classes of MASs considered in the previous subsection is studied. Two additional sufficient conditions are given. The first condition ensures that the consensus state is an equilibrium manifold for the MAS. The second condition is based on the inference graph $\mathcal{G}(f)$; it requires that there must exists a globally reachable node in graph $\mathcal{G}(f)$ and it implies that the consensus state manifold becomes globally asymptotically stable for the MAS. These two conditions applied to MASs as in eq. (4.1) satisfying conditions of Theorem 4.2.1 guarantee that the MAS asymptotically reaches the consensus state.

Before stating the main result, we need to prove some intermediate lemmas. The first important lemma establishes row-stochasticity of the Jacobian matrix of a map evaluated at consensus points if the consensus manifold contains only fixed points. The second lemma establishes that if additionally there exists a fixed point outside the consensus manifold then there exists a consensus point at which the Jacobian has a unitary eigenvalue $\lambda = 1$ with multiplicity strictly greater than one.

Lemma 4.2.3 *Let a map $f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be differentiable. If the set of fixed points $F(f)$ of map f satisfies $F(f) \supseteq \{c\mathbb{1}, \quad c \in \mathbb{R}_+\}$, i.e., the set of fixed points contains all positive consensus states, then the Jacobian matrix J_f of map f computed at a consensus point $c\mathbb{1}$ is row-stochastic, i.e.,*

$$J_f(c\mathbb{1})\mathbb{1} = \mathbb{1} \quad \forall c \in \mathbb{R}_+ .$$

Proof. Since f is differentiable, one can apply directly the definition of directional derivative in a point $x \in \mathbb{R}_+^n$ along a vector $v \in \mathbb{R}^n$ obtaining

$$J_f(x)v = \lim_{h\to 0} \frac{f(x+hv) - f(x)}{h} .$$

Evaluating this expression in a consensus point $x = c\mathbb{1} \in F(f)$ and along the direction $v = \mathbb{1}$ (which is an invariant direction of f) it follows

$$\begin{aligned} J_f(c\mathbb{1})\mathbb{1} &= \lim_{h\to 0} \frac{f(c\mathbb{1} + h\mathbb{1}) - f(c\mathbb{1})}{h} , \\ &= \lim_{h\to 0} \frac{\cancel{c\mathbb{1}} + h\mathbb{1} - \cancel{c\mathbb{1}}}{h} = \mathbb{1} , \end{aligned}$$

thus proving the statement. □

Lemma 4.2.4 *Let a map $f : \mathbb{R}_+^n \to \mathbb{R}_+^n$ be type-K order-preserving and sub-homogeneous. Let the set of fixed points $F(f)$ of map f satisfies $F(f) \supseteq \{c\mathbb{1}, \quad c \in \mathbb{R}_+\}$, i.e., the set of fixed points contains all positive consensus states. If there exists a fixed point $\bar{x} \in \mathbb{R}_+^n$ such that*

$$\bar{x} \neq c\mathbb{1} , \quad \forall c \in \mathbb{R}_+$$

then there exists $\bar{c}(\bar{x}) > 0$ such that the Jacobian matrix $J_f(\bar{c}(\bar{x})\mathbb{1})$ of map f computed at $\bar{c}(\bar{x})\mathbb{1}$ has an eigenvalue $\lambda = 1$ with multiplicity strictly greater than one.

Proof. Let $\bar{x} = [\bar{x}_1, \ldots, \bar{x}_n]^\top \in \mathbb{R}_+^n$ be a fixed point of map f and let $c_1, c_2 \in \mathbb{R}_+$ be such that

$$\begin{aligned} c_1 &= \min_{i=1,\ldots,n} \bar{x}_i , \\ c_2 &= \max_{i=1,\ldots,n} \bar{x}_i . \end{aligned}$$

By defining three sets as follows,

$$\begin{aligned} I_{min}(\bar{x}) &= \{i : \bar{x}_i = c_1\}, \\ I_{max}(\bar{x}) &= \{i : \bar{x}_i = c_2\}, \\ I(\bar{x}) &= \{i : \bar{x}_i \neq c_1, c_2\}, \end{aligned}$$

consider a point y such that the i-th component is defined by

$$y_i = \begin{cases} c_1 & \text{if } i \in I_{min}(\bar{x}) \\ c_3 & \text{otherwise} \end{cases} \tag{4.3}$$

and such that

$$c_1\mathbb{1} \lneq y \lneq \bar{x} \lneq c_2\mathbb{1} \,. \tag{4.4}$$

By (4.3) and (4.4) it follows that

$$y \leq c_3\mathbb{1}. \tag{4.5}$$

- Since map f is type-K order-preserving, from (4.4) it follows $c_1 \leq f_i(y) \leq \bar{x}_i$ and from (4.5) $f_i(y) \leq c_3$ for $i = 1, \ldots, n$. For $i \in I_{min}(\bar{x})$, by definition $\bar{x}_i = c_1$ and thus $f_i(y) = c_1$, otherwise for $i \in I(\bar{x}) \cup I_{max}(\bar{x})$ by (4.4) $\bar{x}_i \geq y_i = c_3$ and it follows $c_1 \leq f_i(y) \leq c_3$. Therefore, it holds

$$f(y) \leq y. \tag{4.6}$$

- Since f is order-preserving and sub-homogeneous, then f is non-expansive under the Thompson's metric (see Definition 3.4.2) by Proposition 3.4.1. Now, by exploiting the definition of non-expansive map, an upper bound to $d_T(\bar{x}, f(y))$ is computed. It holds

$$d_T(\bar{x}, f(y)) \leq d_T(\bar{x}, y) = \log\left(\max\left\{M(\bar{x}/y), M(y/\bar{x})\right\}\right) \tag{4.7}$$

where

$$M(\bar{x}/y) = \max_i \frac{y_i}{x_i} = 1,$$
$$M(y/\bar{x}) = \max_i \frac{x_i}{y_i} \leq \frac{c_2}{c_3}.$$

Since $c_2 \geq c_3$, then

$$d_T(\bar{x}, f(y)) \leq \log\left(\frac{c_2}{c_3}\right). \tag{4.8}$$

On the other hand

$$d_T(\bar{x}, f(y)) = \log\left(\max\left\{M(\bar{x}/f(y)), M(f(y)/\bar{x})\right\}\right)$$

where

$$M(\bar{x}/f(y)) = \max_i \frac{f_i(y)}{x_i} = 1,$$
$$M(f(y)/\bar{x}) = \max_i \frac{x_i}{f_i(y)} \geq \frac{c_2}{c_3}.$$

Since $c_2 \geq c_3$ then

$$d_T(\bar{x}, f(y)) \geq \log\left(\frac{c_2}{c_3}\right). \tag{4.9}$$

By the upperbound (4.8) and the lowerbound (4.9) it follows

$$d_T(\bar{x}, f(y)) = \log\left(\max_i \frac{x_i}{f_i(y)}\right) = \log\left(\frac{c_2}{c_3}\right).$$

Therefore, it holds

$$\frac{c_3}{c_2}\bar{x}_i \leq f_i(y) \leq c_3, \quad i = 1, \ldots, n\,. \tag{4.10}$$

Consider now the iterative behavior of f with initial point y: due to Theorem 3.5.3, for each component f_i of f it holds $\lim_{k\to\infty} f_i^k(y) = \bar{y}_i$. Three cases may occur:

1. If $i \in I_{min}(\bar{x})$ then $\bar{x}_i = c_1$ and by (4.4) it follows $f_i(y) = c_1$.
2. If $i \in I_{max}(\bar{x})$ then $\bar{x}_i = c_2$ and by (4.10) it follows $f_i(y) = c_3$.
3. If $i \in I(\bar{x})$, by (4.6) two cases may occur:
   a) There exists $k^* > 0$ such that $f_i^{k^*}(y) < y_i$. In this case, by type-K order-preservation it holds that

   $$f_i^k(y) < f_i^{k-1}(y) \quad \forall k \geq k^* + 1$$

   and therefore

   $$\lim_{k\to\infty} f_i^k(y) = c_1.$$

   b) Otherwise $f_i^k(y) = f^{k-1}(y) \quad \forall k > 0$ and therefore

   $$\lim_{k\to\infty} f_i^k(y) = y_i = c_3.$$

These consideration can be summarized as follows.

$$\bar{y}_i = \begin{cases} c_1 & \text{if } i \in I_{min}(\bar{x}), \\ c_3 & \text{if } i \in I_{max}(\bar{x}), \\ c_1 & \text{if } i \in I(\bar{x}) \text{ and} \\ & \exists k^* : f_i^{k^*}(y) < f_i^{k^*-1}(y), \\ c_3 & \text{otherwise.} \end{cases} \tag{4.11}$$

So far it has been proved in (4.11) that for any fixed point $\bar{x}$ different from a consensus point $c\mathbb{1}$ there exists a fixed point $\bar{y}$ with elements corresponding to either c_1 or c_3 and such that $I(\bar{y}) = \emptyset$. Last step in the proof is to consider a point z such that its i-th component is defined as follows

$$z_i = \begin{cases} c_1 & \text{if } i \in I_{min}(\bar{y}) \\ c_4 & \text{if } i \in I_{max}(\bar{y}) \end{cases} \tag{4.12}$$

with $c_4 \in [c_1, c_3]$. By (4.11) and (4.12), one can conclude that z is fixed point, i.e., $f(z) = z$, for all values of c_4 in the interval $c_4 \in [c_1, c_3]$. Now, let $v(\bar{x})$ be a vector such that

$$v_i(\bar{x}) = \begin{cases} 0 & \text{if } i \in I_{min}(\bar{x}) \\ 1 & \text{if } i \in I_{max}(\bar{x}) \\ 0 \; or \; 1 & \text{if } i \in I(\bar{x}) \end{cases}, \tag{4.13}$$

Thus, by (4.13) the point $c_1\mathbb{1} + hv(\bar{x})$ is a fixed point of map f for all $h \in [0, c_3 - c_1]$. Thus, it follows that

$$f(c_1\mathbb{1} + hv(\bar{x})) = c_1\mathbb{1} + hv \,, \quad h \in [0, c_3 - c_1] \,.$$

Since $v(\bar{x}) \neq \mathbb{1}$, it holds (by reasoning along the lines of Lemma 4.2.3) that the Jacobian of map f computed at $c_1\mathbb{1}$ has a right eigenvector equal to $v(\bar{x})$, i.e., $J_f(c_1\mathbb{1})v(\bar{x}) = v(\bar{x})$. By Lemma 4.2.3 it holds that the Jacobian of f satisfies $J_f(c\mathbb{1})\mathbb{1} = \mathbb{1}$ for all $c > 0$. Thus, if there exists a fixed point $\bar{x} \neq c\mathbb{1}$ then there exists $\bar{c}(\bar{x}) = \min_{i=1,\dots,n} \bar{x}_i = c_1$ such that matrix $J_f(\bar{c}(\bar{x})\mathbb{1})$ has a unitary eigenvalue with multiplicity strictly greater than one, thus proving the statement of this lemma. □

By means of the two above lemmas, we are ready to prove by contradiction that if the set of equilibrium points of a dynamical systems, satisfying the properties of type-K order-preservation and subhomogeneity, contains the consensus points, then no other equilibrium points may exist. This proves the convergence of all trajectories of a MAS to a consensus state by exploiting Theorem 4.2.1.

Theorem 4.2.5 *Consider n agents evolving according to*

$$x_i(k+1) = f_i(x_i(k), x_j(k) : j \in \mathcal{N}_i).$$

If the set of differentiable local interaction rules $f_i : \mathbb{R}^n_+ \to \mathbb{R}_+$, with $i = 1, \dots, n$, satisfies the next conditions:

(i) $\partial f_i / \partial x_i > 0$ *and* $\partial f_i / \partial x_j \geq 0$ *for* $i \neq j$;
(ii) $\alpha f_i(x) \leq f_i(\alpha x)$ *for all* $\alpha \in [0, 1]$ *and* $x \in \mathbb{R}^n_+$;
(iii) $f_i(x) = x_i$ *if* $x_i = x_j$ *for all* $j \in \mathcal{N}_i$;

(iv) The inference graph $\mathcal{G}(f)$ has a globally reachable node;

then, the MAS asymptotically converges to a consensus state for any initial state $x(0) \in \mathbb{R}^n_+$.

Proof. The proof starts by recalling that conditions (i)-(ii) are equivalent to the properties of type-K order-preservation and plus-subhomogeneity of the map f, as shown in the proof of Theorem 4.2.2. By Theorem 4.2.2, one knows that for any initial conditions $x \in \mathbb{R}^n$, the MAS converge to its set of equilibrium points $F(f)$.

Now, we are going to show that the additional conditions $(iii)-(iv)$ restrict the equilibrium point set $F(f)$ to the consensus space

$$\mathscr{C} = \{\alpha\mathbb{1} : \alpha \in \mathbb{R}\},$$

which will complete the proof. Condition (iii) implies that the consensus space $c\mathbb{1}$ is a subset of the set of fixed points $F(f)$ of map f, i.e.,

$$F(f) \supseteq \{c\mathbb{1} : c \in \mathbb{R}_+\}.$$

By Lemma 4.2.3 the Jacobian matrix $J_f(c\mathbb{1})$ evaluated at a consensus point is row-stochastic, i.e., $J_f(c\mathbb{1})\mathbb{1} = \mathbb{1}$. Clearly, it holds that $\mathcal{G}(f) = \mathcal{G}(J_f(c\mathbb{1}))$. Thus $\mathcal{G}(J_f(c\mathbb{1}))$ has a globally reachable node by hypothesis and is aperiodic because condition (ii) ensures a self-loop at each node.

Assume now that there exists a fixed point $\bar{x} \neq c\mathbb{1}$ with $c \in \mathbb{R}^n_+$. Then, by Lemma 4.2.4 the Jacobian matrix $J_f(c\mathbb{1})$ has a unitary eigenvalue $\lambda = 1$ with multiplicity strictly greater than one. On the other hand, it is well known (e.g., see Theorem 5.1 in [32]) that if $\mathcal{G}(J_f(c\mathbb{1}))$ has a globally reachable node and is aperiodic then $J_f(c\mathbb{1})$ has a simple unitary eigenvalue with corresponding eigenvector equal to $\mathbb{1}$, unique up to a scaling factor c. This is a contradiction, therefore it does not exist a fixed point $\bar{x}$ such that $\bar{x} \neq c\mathbb{1}$ with $c > 0$. Therefore, the set of fixed points of map f satisfies

$$F(f) = \{c\mathbb{1},\ \ c \in \mathbb{R}_+\}.$$

Summarizing, if conditions (a)-(b) are satisfied, then by Theorem 4.2.1 the MAS converges to its set of fixed points $F(f)$. If also (c) is satisfied, the $F(f)$ contains only consensus points and thus the MAS in converges to a consensus state for all $x \in \mathbb{R}^n_+$. □

[32] Bullo (2018), *Lectures on Network Systems*.

A similar result is given for MASs satisfying the properties of type-K order-preservation and plus-homogeneity. Also in this case, if the consensus subspace contains only equilibrium points, then no other equilibrium points exist. This fact is shown in the next theorem.

Theorem 4.2.6 *Consider n agents evolving according to*

$$x_i(k+1) = f_i(x_i(k), x_j(k) : j \in \mathcal{N}_i).$$

If the set of differentiable local interaction rules $f_i : \mathbb{R}^n \to \mathbb{R}$, with $i = 1, \ldots, n$ and $\ell = 1, \ldots, d$, satisfies the next conditions:

(i) $\partial f_i / \partial x_i > 0$ *and* $\partial f_i / \partial x_j \geq 0$ *for* $i \neq j$;
(ii) $f_i(x + \alpha \mathbb{1}) = f_i(x) + \alpha$ *for any* $\alpha \in \mathbb{R}$;
(iii) $f_i(\mathbb{0}) = 0$;
(iv) *Inference graph* $\mathcal{G}(f)$ *has a globally reachable node;*

then, the MAS asymptotically converges to a consensus state for any initial state $x(0) \in \mathbb{R}^n$.

Proof. The proof starts by establishing the relations between properties $(i) - (iv)$ and the following:

(a) f is type-K order-preserving;
(b) f is subhomogeneous;
(c) $F(f) = \{c\mathbb{1} : c \in \mathbb{R}_+\}$.

Each relation is proven next:

1. $[(i) \Leftrightarrow (a)]$ See Proposition 3.2.1 (Kamke-like condition).
2. $[(ii) \Leftrightarrow (b)]$ See Proof of Theorem 4.2.2.
3. $[(i - iv) \Rightarrow (c)]$ The proof of this implication is given below.

Exploiting the result in Theorem 4.2.1 one knows that for any initial conditions $x \in \mathbb{R}^n$, the MAS converge to its set of equilibrium points $F(f)$. It is shown that conditions $(iii) - (iv)$ ensure that the fixed point set $F(f)$ coincide with the consensus space

$$\mathcal{C} = \{\alpha \mathbb{1} : \alpha \in \mathbb{R}\},$$

which will complete the proof. Condition (iii) implies that the origin is a fixed point of map $f(\cdot)$, thus $f(\mathbf{0}) = \mathbf{0}$. By plus-homogeneity one derive that all consensus points are fixed points of map $f(\cdot)$, as follows

$$\begin{aligned} f(\mathbf{0} + \alpha \mathbb{1}) &= f(\mathbf{0}) + \alpha \mathbb{1}, & \forall \alpha \in \mathbb{R}, \\ f(\alpha \mathbb{1}) &= \alpha \mathbb{1}, & \forall \alpha \in \mathbb{R}. \end{aligned}$$

Therefore, the fixed point set of map $f(\cdot)$ contains all consensus points, i.e., $F(f) \supseteq \mathcal{C}$. Following the same reasoning of the proof of Theorem 3.5.4, the log-exp transform $g : \text{int}(\mathbb{R}^n) \to \text{int}(\mathbb{R}^n)$ of $f(\cdot)$ given by $g = \exp \circ f \circ \log$ for $z \in \text{int}(\mathbb{R}^n)$ is a type-K order-preserving and homogeneous map. By condition (iv) and the bijective relation between $\mathbb{R}^n$ and $\text{int}(\mathbb{R}_+)$ given by the exponential and logarithmic functions, it follows that:

- $F(g) \supseteq \mathscr{C}_+ = \{\alpha\mathbb{1} : \alpha \in \text{int}(\mathbb{R}_+)\}$, since $F(f) \subseteq \mathscr{C}$.
- $\mathscr{G}(g)$ has a globally reachable node, since $\mathscr{G}(f)$ has a globally reachable node.

By Lemma 4.2.3 the Jacobian matrix J_g of map $g(\cdot)$ is stochastic at the consensus points. By the definition of inference graph, it holds that $\mathscr{G}(g) = \mathscr{G}(J_g(c\mathbb{1}))$. Thus $\mathscr{G}(J_g(c\mathbb{1}))$ has a globally reachable node by condition (iv) and is aperiodic because condition (ii) ensures a self-loop at each node. If there exists a fixed point $\bar{z} \neq \alpha\mathbb{1}$, then by Lemma 4.2.4 the Jacobian matrix $J_f(c\mathbb{1})$ has a unitary eigenvalue with multiplicity strictly greater than one. On the other hand, it is well known (e.g., see Theorem 5.1 in [32]) that if $\mathscr{G}(J_g(\alpha\mathbb{1}))$ has a globally reachable node and is aperiodic then $J_g(c\mathbb{1})$ has a simple unitary eigenvalue with corresponding eigenvector equal to $\mathbb{1}$, unique up to a scaling factor c. This is a contradiction, therefore it does not exist a fixed point $\bar{z}$ such that $\bar{z} \neq \alpha\mathbb{1}$. Therefore,

[32] Bullo (2018), *Lectures on Network Systems*.

$$F(g) = \mathscr{C}_+ \quad \Leftrightarrow \quad F(f) = \mathscr{C}.$$

Summarizing, if conditions (a)-(b) are satisfied, then by Theorem 4.2.2 the MAS converges to its set of fixed points $F(f)$. If also (c) is satisfied, the $F(f)$ contains only consensus points and thus the MAS in converges to a consensus state for all $x \in \mathbb{R}^n_+$. □

4.3 Application to epidemics over networks

Propagation phenomena appear in numerous disciplines. One of the approach in the analysis of propagation models is based on the mean-field approximation of Markov-chain models and algebraic graph theory. The main assumption of these models is the knowledge of the local propagation parameters, which allows to relate the dynamical behavior of the propagation process to some global parameters of the network but without the need to know the network itself. One of the main advantages of such an approach is that, by representing the network as an adjacency matrix, well-established theorems in matrix analysis and dynamical systems can be applied to the analysis of some sophisticated behavior of the propagation processes. However, when the dynamics becomes nonlinear, these results can not be employed and the analysis become more complicated.

In this section the focus is on a specific epidemic propagation model called the susceptible-infected-susceptible (SIS) model. The population is assumed to be partitioned in several groups (e.g., the sex, the age and so on) and consider the spread of the disease be different between different groups. In particular, the discrete-time network SIS model given in [8] is considered, which appears to be the first to revisit and formally reproduce, for the discrete-time case, the earlier results by Lajmanovich et al. [130]. The aim of this section is to provide a convergence analysis of this model within the theory developed in Chapters 3-4.

[8] Allen (1994), 'Some discrete-time SI, SIR, and SIS epidemic models.'

[130] Lajmanovich and Yorke (1976), 'A deterministic model for gonorrhea in a nonhomogeneous population'.

Most of the literature considers continuous approximations of these models, due to their mathematical tractability. However, it is clear that the system under consideration is intrinsically discrete and a continuous approximation may lead to loss of information. In fact, behavior of coherent discrete-time models is not as well-behaved as their continuous approximations, due to eventual periodic and chaotic behavior, which are absent in the continuous models. Here the focus is on the discrete-time SIS model presented in the recent work [233], which places a network twist [183] on the earlier work [8].

[233] Wang et al. (2019), 'Dynamical analysis of a discrete-time SIS epidemic model on complex networks'.

[183] Pastor-Satorras and Vespignani (2001), 'Epidemic spreading in scale-free networks'.

The epidemic is assumed to propagate over a graph $\mathcal{G} = (V, E)$. Nodes of G can be interpreted as homogeneous groups of individuals and edges can be interpreted as the connection between different groups. Each group is subdivided according to susceptible and infectious. The state $x_i \in [0, 1]$ of the i-th group denotes the portion of susceptible people in that group, while $y_i = 1 - x_i$ denotes the portion of infected people. Individuals can be cured and reinfected many times and there is not an immune group. The dynamics of susceptible people x_i and of infected people y_i in each group $i \in V$ are given by

$$x_i(k+1) = x_i(k)(1 - h \sum_{j \in \mathcal{N}_i^\circ} \alpha_{ij} y_j(k)) + \gamma_i h y_i(k) \tag{4.14}$$

$$y_i(k+1) = y_i(k)(1 - \gamma_i h) + h x_i(k) \sum_{j \in \mathcal{N}_i^\circ} \alpha_{ij} y_j(k) \tag{4.15}$$

where α_{ij} denotes the infection rate between groups i and j, γ_i

denotes the recovery rate of group i, h denotes the sampling time and $\mathcal{N}_i^\circ = \mathcal{N}_i \cup \{i\}$. Assuming a suitable choice of the parameters guaranteeing the boundedness of the solutions, there can exists either a disease-free equilibrium or an endemic equilibrium. The stability analysis of these equilibrium points is not trivial and global results exists only for the disease-free equilibrium [8]. In fact, the endemic equilibrium may be unstable, giving rise to periodicity or chaos, as it is shown in the following example.

Let the population be divided into two groups, males x_1 and females x_2, and consider the spread of a sexual disease. Letting $A = \{\alpha_{ij}\}$, three cases are considered:

1. Heterosexual contacts with $A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$ and $h = 0.2$.
2. Bisexual contacts with $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ and $h = 0.4$.
3. Homosexual contacts with $A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ and $h = 1.1$.

In all these cases the recovery rate for both population is taken equal to $\gamma_i = 1.5$ for any $i \in \{1, 2\}$, and the initial condition is taken equal to $y_1 = 0.2.$ and $y_2 = 0.8$. According to [8], the choice of the parameters ensures the boundedness of the solutions. In Figure 4.1 the state evolution of the system in these three cases showing the emergence of periodicity and chaos are given. In particular, bisexual contacts give the periodic evolution in the top figure, heterosexual contacts give the stable evolution toward an endemic equilibrium in the middle figure and homosexual contacts gives the chaotic evolution (for males) and stable evolution toward a disease-free situation (for females) in the bottom figure. The dynamics of the discrete-time SIS epidemic on complex networks is complicated under the effects the infection rates α_{ij}, the recovery rates γ_i, and the time step-size h. Comparing the discrete SIS model on complex networks with the continuous counterpart, one notice that if R_0, the endemic equilibrium is stable for the continuous model while may be unstable (or even chaotic) for the discrete model. This confirms that the discrete model on complex networks has more complex dynamical behaviors than the continuous counterpart.

In the next theorem sufficient conditions on the model (4.14)-(4.15) are given ensuring convergence to an equilibrium, either disease free or endemic, while preventing periodic and chaotic behaviors.

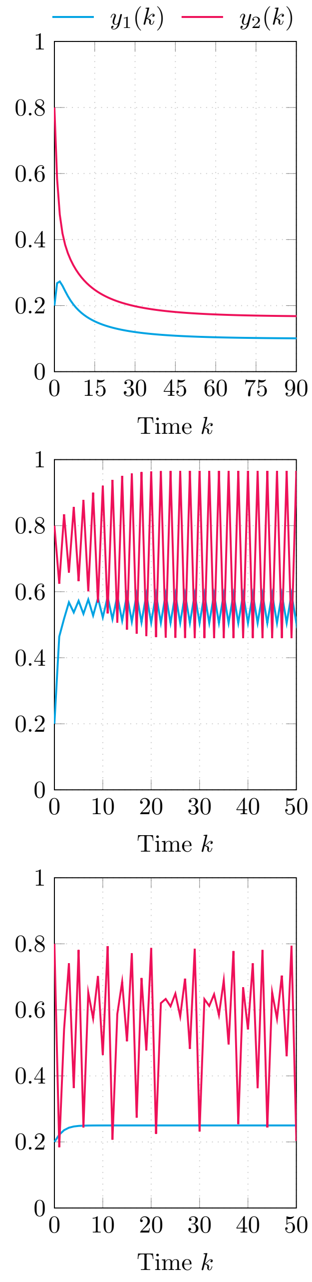

Figure 4.1: Spread of an infectious disease.

Theorem 4.3.1 *Consider the discrete-time SIS network model with n groups with dynamics as in* (4.14)-(4.15). *If the following conditions on the network parameters is satisfied*

$$h \sum_{j\in\mathcal{N}_i^\circ} \alpha_{ij} < 1 - h\gamma_i, \quad i = 1, \ldots, n. \tag{4.16}$$

then each equilibrium of the system (either disease free or endemic) is stable and no periodic or chaotic trajectories exist, for any initial condition $x \in [0,1]^n$.

Proof. The proof consists in showing that the system of infected people, rewritten as

$$y_i(k+1) = f_i(y(k)) = y_i(k)(1-\gamma_i h) + h(1-y_i(k)) \sum_{j\in\mathcal{N}_i^\circ} \alpha_{ij} y_j(k)$$

satisfies conditions $(i) - (ii)$ of Theorem 4.2.1. Condition (i) of Theorem 4.2.1 is satisfied, i.e., the system is type-K order-preserving if the following holds

$$h \sum_{j\in\mathcal{N}_i^\circ} \alpha_{ij} < 1 - h\gamma_i, \tag{4.17}$$

in fact one can compute

$$\begin{aligned}
\frac{\partial}{\partial y_i} f_i &= 1 - \gamma_i h - h \sum_{j\in\mathcal{N}_i} \alpha_{ij} y_j(k) + h\alpha_{ii} - 2h\alpha_{ii} y_i(k) \\
&\geq 1 - \gamma_i h - h \sum_{j\in\mathcal{N}_i} \alpha_{ij} - h\alpha_{ii} \\
&\geq 1 - \gamma_i h - h \sum_{j\in\mathcal{N}_i^\circ} \alpha_{ij} > 0 \\
\frac{\partial}{\partial y_j} f_i &= h(1-y_i(k))\alpha_{ij} \geq 0.
\end{aligned}$$

It is easy to derive that if (4.17) is satisfied, then solutions $x(k)$ and $y(k)$ of the system (4.15) remains enclosed in $[0,1]^n$ (cfr, Lemma 3 in [8]). Condition (ii) of Theorem 4.2.1 is always satisfied, i.e., the

[8] Allen (1994), 'Some discrete-time SI, SIR, and SIS epidemic models.'

system is subhomogeneous, in fact one can compute

$$\begin{aligned}
\alpha f_i(y) &\leq f_i(\alpha y)\\
\alpha h(1-y_i(k))\sum_{j\in\mathcal{N}_i^\circ}\alpha_{ij}y_j(k) &\leq h(1-\alpha y_i(k))\sum_{j\in\mathcal{N}_i^\circ}\alpha_{ij}\alpha y_j(k)\\
\alpha h(1-y_i(k))\sum_{j\in\mathcal{N}_i^\circ}\alpha_{ij}y_j(k) &\leq \alpha h(1-\alpha y_i(k))\sum_{j\in\mathcal{N}_i^\circ}\alpha_{ij}y_j(k)\\
-\alpha h(y_i(k))\sum_{j\in\mathcal{N}_i^\circ}\alpha_{ij}y_j(k) &\leq -\alpha^2 h(y_i(k))\sum_{j\in\mathcal{N}_i^\circ}\alpha_{ij}y_j(k)\\
-\alpha &\leq -\alpha^2\\
1 &\geq \alpha.
\end{aligned}$$

It can be verified that the disease-free equilibrium $\bar{y} = \mathbb{0}$ is always an equilibrium point of the system. Let us now consider the translated system $z_i(k) = y_i(k) + c$ with $c \in \mathbb{R}_+$, i.e., $z(k+1) = f_i(z_i(k) - c) + c$. It is clear that the point $\bar{z} = c\mathbb{1}$ is a positive equilibrium point of the translated system and that conditions (i)-(ii) of Theorem 4.2.1 still hold under condition in eq. (4.17). Thus, we conclude that the MAS converges to an equilibrium point for all $x \in [0,1]^n$, completing the proof. □

The condition provided in the above theorem is only sufficient and not necessary. Let us discuss the examples given before in the light of our theorem:

1. Simulation for heterosexual contacts with $A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$ and $h = 0.2$ is shown at the top of Figure 4.1. Condition (4.16) is verified since $h(\alpha_{11} + \alpha_{12}) = 0.2$ and $h(\alpha_{21} + \alpha_{22}) = 0.6$ which are both strictly lesser than $1 - h\gamma_1 = 1 - h\gamma_2 = 0.7$. Thus, periodic and chaotic trajectories are avoided and the system converges to an equilibrium which is endemic due to the the particular choice of the initial state.
2. Simulation for bisexual contacts with $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ and $h = 0.4$ is shown in the middle of Figure 4.1. Condition (4.16) is not verified since $h(\alpha_{11}+\alpha_{12}) = h(\alpha_{21}+\alpha_{22}) = 2$ which is greater than $1 - \gamma_i h$. In fact, it is the case that the system shows a periodic behavior.
3. Simulation for homosexual contacts with $A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ and $h = 1.1$ is shown in the bottom of Figure 4.1. Condition (4.16) is not verified since $h(\alpha_{11} + \alpha_{12}) = 0.5h(\alpha_{21} + \alpha_{22}) = 2.2$ which is grater than $1 - \gamma_i h$. In fact, it is the case that the system shows a chaotic behavior.

As a final example, we simulate the spread of an epidemic in a network of 8 groups connected according to the graph $\mathcal{G}_1$ depicted in Fig. 4.2. The infection rates are chosen all equal to $\alpha_{ij} = 1$ for $i, j \in V$, the recovery rates are chosen all equal $\gamma_i = 1.5$ for $i \in V$ and the sampling time is chosen equal to $h = 0.2$; the choice of the parameters satisfies Theorem 4.3.1. Simulations with a random initial condition is given in Fig.4.3. It can be seen the system asymptotically reaches a steady state, which is an endemic equilibrium.

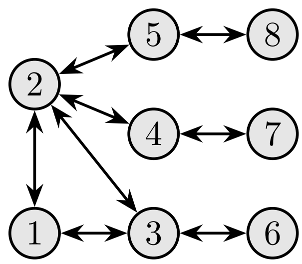

Figure 4.2: graph $\mathcal{G}_1$.

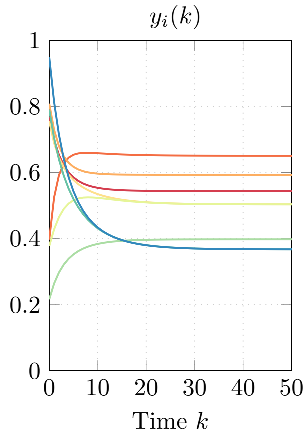

Figure 4.3: Evolution of the system with parameters: $h = \alpha_{ij} = \varepsilon_i = 0.5$ for all i and $j \in \mathcal{N}_i$.

4.4 Application to max-consensus

In this section a novel protocol is presented to solve the max-consensus problem in MASs, i.e., the problem of steering all agents to the maximum value among all initial agents' states. The aim of this section is to provide a convergence analysis of this protocol within the theory developed in Chapters 3-4. This result is given in the next theorem.

Theorem 4.4.1 *Consider n agents evolving according to*

$$x_i(k+1) = f_i(x(k)) = x_i(k) + h_i \sum_{j \in \mathcal{N}_i} \Big(d_{ji}(k) + |d_{ji}(k)| \Big), \quad (4.18)$$

with $d_{ji} = x_j(k) - x_i(k)$ and a graph $\mathcal{G}$ with a globally reachable node. If

$$h_i < \frac{1}{2|\mathcal{N}_i|} \quad \forall i \in V, \quad (4.19)$$

then, for any initial state $x(0) \in \mathbb{R}^n_+$, the MAS asymptotically converges to the maximum initial state, i.e.,

$$\lim_{k \to \infty} x_i(k) = \max_{j \in V} x_j(0).$$

Proof. The proof consists in showing that the system satisfies conditions $(i) - (iv)$ of Theorem 4.2.5, thus proving its convergence to a consensus state, and then proving that such a consensus state corresponds to the maximum among all the initial agents' state.

Condition (i) of Theorem 4.2.5 is analyzed as follows

$$\begin{aligned}\frac{\partial f_i}{\partial x_i} &= 1 + h_i \left[-|\mathcal{N}_i| - \sum_{j \in \mathcal{N}_i} \operatorname{sign}(x_j(k) - x_i(k)) \right] \\ &\geq -2|\mathcal{N}_i| h_i \qquad \text{if } x_j(k) > x_i(k), \forall j \in \mathcal{N}_i \\ &> 0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial f_i}{\partial x_j} &= 0 + h_i \left[1 + \operatorname{sign}(x_j(k) - x_i(k))\right] \\ &\geq h_i \left[1 - 1\right] \qquad \text{if } x_j(k) < x_i(k) \forall j \in \mathcal{N}_i \\ &\geq 0\,.\end{aligned}$$

Thus, condition (i) holds if and only if (4.19) holds. Condition (ii) of Theorem 4.2.5 holds for any $x \in \mathbb{R}^n_{\geq 0}$. This follows from the fact that the only nonlinear term of f_i is $|\cdot|$ for which it holds $|\alpha x| = \alpha |x|$. Therefore, $\alpha f_i(x) = f_i(\alpha x)$, which implies sub-homogeneity. Condition (iii) of Theorem 4.2.5 is satisfied since $\bar{x} = c\mathbb{1}$ with $c > 0$ is a positive fixed point. Condition (iv) of Theorem 4.2.5 is satisfied since the graph is assumed to contain a globally reachable node.

Thus, if (4.19) holds, then all conditions of Theorem 4.2.5 are satisfied, and one concludes that the MAS in (4.18) converges to a consensus state.

It is straightforward to show that the consensus state reached by a MAS in (4.18) is exactly the maximum among the initial states of all agents. In fact, without loss of generality, suppose that node 1 has the maximum value, i.e., $x_1 \geq x_i$ for all $i \in V$. In this case, $x_1(k+1) = x_1(k)$, since the second term in (4.18) is 0. Since agent i keeps its value for all $k \geq 0$, it is clear that the MAS in (4.18) reaches consensus on the maximum value among all initial states, completing the proof. □

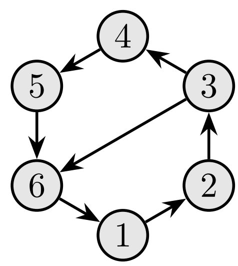

Figure 4.4: Graph $\mathcal{G}_2$.

Fig.4.5 shows the evolution of a MAS with $n = 6$ agents connected according to the graph $\mathcal{G}_2$ depicted in Fig. 4.4. The choice of the parameter $h = 0.1$ for $j \in \mathcal{N}_i$ and $i \in \{1, 2\}$ satisfies Theorem 4.4.1. Thus, the system asymptotically converges to the maximum among the initial state of the agents.

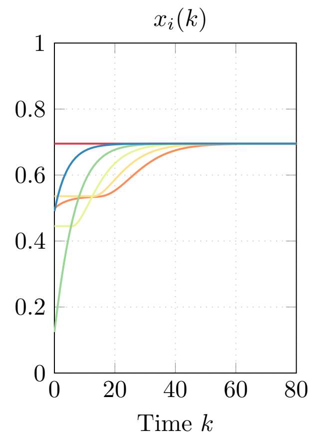

Figure 4.5: Evolution of the system with parameters: $h_i = 0.1$.

5 Stability and consensus of continuous-time MASs

In this chapter MASs in which the agents are single continuous-time integrators are considered. The dynamics of each agent is described by

$$\dot{x}_i(t) = f_i(x_i(t), x_j(t) : j \in \mathcal{N}_i). \tag{5.1}$$

where $x_i(t) \in \mathbb{R}$ represents the state of the i-th agent at time t. Denoting $x = [x_1, \cdots, x_n]^\intercal$ the state of the MAS, with $n \in \mathbb{N}$ being the number of the agents, the global dynamics is written as

$$\dot{x}(t) = f(x(t)). \tag{5.2}$$

In a continuous-time framework, the properties of order-preservation and homogeneity discussed at length in the previous chapters, translate into *monotonicity* and *translation invariance*. These properties regard the flow generated by a vector field and not anymore the sequence of points generated by a map. The focus of this chapter is on the following class:

- Type-K monotone and translation invariant flows in $\mathbb{R}^n$.

In Section 5.1 the relative literature is reviewed along with a self-contained and coherent definition of these systems and their properties. The first main contribution of this chapter is the convergence analysis of trajectories generated by this class of systems by means of the previous results given in Section 3.5, thus exploiting nonlinear Perron-Frobenius Theory. The related literature and a self-contained and coherent definition of these properties is provided in Section 5.1. The second main contribution is the application of this new mathematical tool to the analysis of the stability and convergence to consensus of MASs. More precisely:

- Sufficient conditions on the local interaction rule of a generic agent are proposed (such a rule can be potentially different for each agent, i.e., heterogeneous) which guarantees that the system falls into the considered class and is stable if an equilibrium point exists;
- Sufficient conditions linking the topology of the network and the structure of the local interaction rules are proposed, which guarantees the achievement of a consensus state.

5.1 Monotone and translation invariant systems

Nonlinear Perron–Frobenius theory is related to monotone dynamical systems theory. In the theory of monotone dynamical systems, emphasis is placed on continuous-time systems being strongly monotone. Pioneering work in this field was done by Hirsch [105] who first showed that if solutions of continuous-time strongly monotone dynamical systems exist and are bounded, then they converge to the set of equilibrium points. An extensive overview of these results was given by Hirsch and Smith [102–104, 106, 210]. Remarkably, under suitable additional assumptions, generic convergence to equilibria can be made global, as in the case of unique equilibrium point [210]. If, however, one relax the strong assumption and only assumes the dynamical system to be monotone, most of the theory is not applicable. In contrast, in nonlinear Perron–Frobenius theory one usually considers discrete-time dynamical systems that are only monotone, but satisfy an additional assumption, such as the various version of homogeneity introduced in Chapter 3.

A number of paper focused on monotone dynamical systems whose vector field possess the property of homogeneity. As in Chapter 4 the dynamics of these systems is usually restricted to the positive orthant $\mathbb{R}^n_+$. Pivotal in the analysis of monotone and homogeneous systems were the extenstions to the nonlinear case of the Perron-Frobenius Theorem [3, 91], which allowed to prove that the vector field on an invariant ray determines the stability properties of the zero solution with respect to initial conditions. Further results in this area include d-stability analysis [149], delay-independency [30]. Generalizations of these results to subhomogeneous vector fields can be found in [29, 73, 251].

In the context of continuous-time systems, the property of plus-homogeneity is known as the property of translation invariance of the solutions. Angeli and Sontag in a series of recent works studied monotone systems from a control perspective [10], their interconnection [11] and multi-stability [12] properties, and possessing a translation invariant property [13, 14]. In particular, in [14] it has been shown that a strongly monotone and translation invariant system with a non-empty set of equilibria, always converges to one of its equilbrium points. This result was generalized to monotone system, getting rid of the strong assumption, by Hu and Jiang [108]. This methodology has far-reaching implications,

[105] Hirsch (1988), 'Stability and convergence in strongly monotone dynamical systems'.

[103] Hirsch (1983), 'Differential equations and convergence almost everywhere in strongly monotone semiflows'.

[104] Hirsch (1985), 'Systems of differential equations that are competitive or cooperative II: Convergence almost everywhere'.

[210] Smith (1996), 'Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems'.

[106] Hirsch and Smith (2003), 'Competitive and cooperative systems: a mini-review'.

[102] Hirsch and Smith (2006), 'Chapter 4 Monotone Dynamical Systems'.

[210] Smith (1996), 'Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems'.

[3] Aeyels (2002), 'Extension of the Perron–Frobenius Theorem to Homogeneous Systems'.

[91] Gaubert and Gunawardena (2004), 'The Perron-Frobenius theorem for homogeneous, monotone functions'.

[149] Mason and Verwoerd (2009), 'Observations on the stability properties of cooperative systems'.

[30] Bokharaie et al. (2010), 'D-stability and delay-independent stability of homogeneous cooperative systems'.

but so far has not received much attention in the literature. In this chapter these results are applied to the analysis of MASs. In particular, conditions on the local interaction rule of an agent are identified in order to ensure that the MAS possesses the above discussed properties, thus ensuring its stability and convergence to consensus as a special case when an additional graph theoretical condition is satisfied.

In the following, the theoretical background about monotone and translation invariant system is provided. This class of system is the continuous-time counterpart of order-preserving and plus-homogeneous systems previously introduced in Chapter 3.

Monotonicity

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field. A solution of a dynamical system (5.1) is denoted as $\varphi(t, x_0)$, where $x_0 \in \mathbb{R}^n$ denotes the initial condition. Monotonicity is a property of the solutions generated by the vector field, requiring that two solutions starting from ordered initial conditions maintain the order over time. Next this property is formalized.

Definition 5.1.1 *A dynamical system $\dot{x}(t) = f(x(t))$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be*

- monotone *if $\forall x, y \in \mathbb{R}^n$ its solutions $\varphi(\cdot)$ satisfy*

$$x \leq y \Rightarrow \varphi(t, x) \leq \varphi(t, y).$$

- strictly monotone *if $\forall x, y \in \mathbb{R}^n$ its solutions $\varphi(\cdot)$ satisfy*

$$x \lneq y \Rightarrow \varphi(t, x) \lneq \varphi(t, y).$$

- strongly monotone *if $\forall x, y \in \mathbb{R}^n$ its solutions $\varphi(\cdot)$ satisfy*

$$x \lneq y \Rightarrow \varphi(t, x) < \varphi(t, y).$$

In addition to these notions of monotonicity, a new notion is introduced that is in between strict and strong monotonicity. This property is denoted as as *type-K monotonicity*.

Definition 5.1.2 *A dynamical system $\dot{x}(t) = f(x(t))$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be type-K monotone if $\forall x, y \in \mathbb{R}^n$ and $x \lneq y$ its solutions $\varphi(\cdot)$ satisfy*

[29] Bokharaie and Mason (2014), 'On delay-independent stability of a class of nonlinear positive time-delay systems'.
[73] Feyzmahdavian et al. (2014), 'Sub-homogeneous positive monotone systems are insensitive to heterogeneous time-varying delays'.
[251] Zhao and Meng (2020), 'Input-to-State Stability Analysis of Sub-homogeneous Cooperative Systems'.
[10] Angeli and Sontag (2003), 'Monotone control systems'.
[11] Angeli and Sontag (2004), 'Interconnections of monotone systems with steady-state characteristics'.
[12] Angeli and Sontag (2004), 'Multi-stability in monotone input/output systems'.
[13] Angeli and Sontag (2006), 'A note on monotone systems with positive translation invariance'.
[14] Angeli and Sontag (2008), 'Translation-invariant monotone systems, and a global convergence result for enzymatic futile cycles'.
[14] Angeli and Sontag (2008), 'Translation-invariant monotone systems, and a global convergence result for enzymatic futile cycles'.
[108] Hu and Jiang (2010), 'Translation-invariant monotone systems, I: autonomous/periodic case'.

$(i)\ x_i = y_i \Rightarrow \varphi_i(t,x) \le \varphi_i(t,y)$,
$(ii)\ x_i < y_i \Rightarrow \varphi_i(t,x) < \varphi_i(t,y)$,

for all $i = 1,\dots,n$, where φ_i is the i-th component of φ.

We remark that the property of monotonicity and its derivations are are the natural counterpart on continuous-time systems of the equivalent property of order-preservation for discrete-time system, presented in Definitions 3.2.2-3.2.3. Usually, to verify monotonicity is not an easy task. For differentiable continuous-time dynamical systems a sufficient condition to ensure monotonicity is the well-known *Kamke condition* [113, 211]. In the following proposition the result of Kamke condition is generalized by proving that it implies type-K monotonicity and not only monotonicity.

[211] Smith (1988), 'Systems of Ordinary Differential Equations Which Generate an Order Preserving Flow. A Survey of Results'.
[113] Kamke (1932), 'Zur Theorie der Systeme gewöhnlicher Differentialgleichungen. II.'

Proposition 5.1.1 *The dynamical system $\dot{x}(t) = f(x(t))$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ is type-K monotone if and only if its Jacobian matrix is Metzler, i.e., if*

$$\partial f_i/\partial x_j > 0 \text{ for } i \neq j\,. \tag{5.3}$$

Proof. Consider two initial conditions $y(0), z(0) \in \mathbb{R}^n$ and the corresponding solutions $\varphi(t, y(0))$ and $\varphi(t, z(0))$ for $t \ge 0$. Let us denote $I \subset \{1,\dots,n\}$. Type-K monotonicity requires that if

$$\begin{aligned} y_i(0) &= z_i(0) && i \in I \\ y_i(0) &< z_i(0) && \text{otherwise} \end{aligned}$$

then

$$\varphi_i(t, y(0)) \le \varphi_i(t, z(0)) \qquad i \in I \tag{5.4}$$

$$\varphi_i(t, y(0)) < \varphi_i(t, z(0)) \qquad \text{otherwise.} \tag{5.5}$$

Let us introduce the one (positive) parameter family of differential equations

$$\dot{x}(t) = g(x(t)) - \frac{1}{n}v, \qquad v \in \text{int}(\mathbb{R}^n_+), n \ge 0$$

with the initial condition $x(0) = z(0) - \frac{1}{n}v$. The solution of this system satisfying the initial condition is denoted by $\phi(t, x(0))$. Function $\phi(\cdot)$ is a solution of the differential inequality $\dot{x}(t) < g(x(t))$ and $x(0) < z(0)$, thus $\phi(t, x(0)) < \varphi(t, z(0))$, cfr Theorem 8 in [43]. Because of the continuous dependence on initial values and parameters, it follows that $\phi(t, x(0)) \to \varphi(t, y(0))$ as $n \to \infty$. Hence,

$$\varphi(t, y(0)) \le \varphi(t, z(0)), \qquad t \ge 0. \tag{5.6}$$

[43] Coppel (1965), *Stability and asymptotic behavior of differential equations*.

Now it is of interest examining under what conditions it may happen that $\varphi_i(t^*, y(0)) = \varphi_i(t^*, z(0))$ for $t^* > 0$. Let $\hat{x} \in \mathbb{R}^{n-1}$ denote the vector formed by the coordinates of x other than x_i, then

$$\dot{x}_i(t) = g_i(x(t)) = g_i(\hat{x}(t), x_i(t)) = G_i(t, x_i).$$

Solutions to this equation are denoted by $\tilde{\varphi}$ and satisfy $\tilde{\varphi}_i(\tau, y(0)) = \varphi_i(t, y(0))$ and $\tilde{\varphi}_i(\tau, z(0)) = \varphi_i(t, z(0))$. At the time $\tau^* = -t^*$ it holds $\tilde{\varphi}_i(\tau^*, y(0)) = \varphi_i(\tau^*, z(0))$ and thus, by following the same reasoning of before, one conclude that for $\tau \in [-t^*, 0]$ or equivalently $t \in [0, t^*]$ it holds

$$\tilde{\varphi}_i(\tau, y(0)) \geq \tilde{\varphi}_i(\tau, z(0)) \quad \Rightarrow \quad \varphi_i(t, y(0)) \geq \varphi_i(t, z(0)).$$

Since condition (5.4) must hold, then $\varphi_i(t^*, y(0)) = \varphi_i(t^*, z(0))$ implies that $\varphi_i(t, y(0)) = \varphi_i(t, z(0))$ for any $t \in [0, t^*]$ and therefore $y_i(0) = z_i(0)$, formally

$$\varphi_i(t, y(0)) = \varphi_i(t, z(0)), \quad \Rightarrow \quad y_i(0) = z_i(0). \tag{5.7}$$

The combination of conditions (5.6) and (5.7), completes the sufficiency part of the proof, i.e., "condition (5.3) $\longrightarrow$ type-K monotonicity". On the other hand, it holds that "type-K monotonicity $\Rightarrow$ monotonicity" and "condition (5.3) $\Leftrightarrow$ monotonicity" as it has been shown in [174, 210]. Therefore, "condition (5.3) $\Leftrightarrow$ type-K monotonicity", completing the proof. □

[174] O'Donoghue et al. (2012), 'On the Kamke–Müller conditions, monotonicity and continuity for bi-modal piecewise-smooth systems'.
[210] Smith (1996), 'Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems'.

Translation invariance

The property of translation invariance is now introduced. This is the continuous-time counterpart of plus-homogeneity for discrete-time systems, , presented in Definition 3.3.2.

Definition 5.1.3 *A dynamical system $\dot{x}(t) = f(x(t))$ with $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be translation invariant if its solutions satisfy*

$$\varphi(t, x + \alpha\mathbb{1}) = \varphi(t, x) + \alpha\mathbb{1}, \quad \forall t \geq 0$$

for all $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

A main feature of type-K monotone systems possessing also translation invariance property is that of being non-expansive under the sup-norm defined.

Definition 5.1.4 *A dynamical system $\dot{x}(t) = f(x(t))$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be non-expansive with respect to a metric $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ if its solutions $\varphi(\cdot)$ satisfy*

$$d(\varphi(t,x), \varphi(t,y)) \leq d(x,y), \quad \forall t \geq 0$$

for all initial conditions $x, y \in \mathbb{R}^n$ and $t \geq 0$.

Proposition 5.1.2 *If a dynamical system $\dot{x}(t) = f(x(t))$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ is type-K monotone and translation invariant then it is non-expansive with respect to the sup-norm.*

Proof. Consider any initial conditions $y, z \in \mathbb{R}^n$ and solutions $\varphi(t,y)$, $\varphi(t,z)$. For any time t, the map $h_t(x) = \varphi(t,x)$ is type-K order preserving and plus-homogeneous since the system is assumed to be type-K monotone and translation invariant. Therefore, by Proposition 3.4.2, it holds

$$\left\| h_t(x) - h_t(y) \right\|_\infty \leq \left\| x - y \right\|_\infty, \quad \forall t \geq 0,$$

completing the proof. □

5.2 Stability and consensus analysis

Before establishing the main results of this section, we recall the definition of inference graph given in Chapter 4.

Definition 5.2.1 *Given a differentiable map $f : X \to X$ with $X \subseteq \mathbb{R}^n$, its inference graph $\mathcal{G}(f) = (V, E)$ has a set of n nodes $V = \{1, \ldots, n\}$ and a set of directed edges $E \subseteq V \times V$. An edge $(i,j) \in E$ from node i to node j exists if*

$$\frac{\partial}{\partial x_j} f_i(x) \neq 0, \qquad \forall x \in X \setminus S,$$

where S is a set of measure zero in X.

The next theorem provides an alternative proof of the convergence of trajectories generated by type-K monotone and translation invariant systems of systems, which was originally established in [108]. The proof is carried out by means of the result given in Section 3.5 and thus exploiting nonlinear Perron-Frobenius Theory.

[108] Hu and Jiang (2010), 'Translation-invariant monotone systems, I: autonomous/periodic case'.

Theorem 5.2.1 *Let a dynamical system $\dot{x}(t) = f(x(t))$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ be type-K monotone and translation invariant. If $f(\cdot)$ has at least one equilibrium point, then it holds that*

$$\lim_{t\to\infty} \varphi(t, x_0) = \bar{x}, \quad \forall x_0 \in \mathbb{R}^n$$

where $\bar{x}$ is an equilibrium point of the system.

Proof. Consider $t = kT$ with the discrete time index $k \in \{0, 1, 2, \dots\}$ and consider the iterative sequence of points generated by the discrete-time system [154]

[154] Mickens (1994), *Nonstandard finite difference models of differential equations.*

$$\hat{x}(k+1) = g(x(k)) = \varphi(T, \hat{x}(k)), \quad \forall x \in \mathbb{R}^n, \tag{5.8}$$

for any $T \in \mathbb{R}$ and let the initial condition be $\hat{x}(0) = x(0)$. It is clear that the trajectory $\mathcal{T}(x, g)$ is entirely contained in the trajectory $\mathcal{T}(x, f)$. Therefore, one conclude that one can study trajectories of the continuous-time system by equivalently studying the trajectories of the discrete-time system (5.8), in fact

$$\lim_{k\to\infty} g^k(x) \equiv \lim_{t\to\infty} \varphi(t, x), \quad \forall x \in \mathbb{R}.$$

The continuous-time system is assumed to be type-K monotone and translation invariant. Due to type-K monotonicity, solutions $\varphi(\cdot)$ are type-K order-preserving maps. Due to translation invariance, solutions $\varphi(\cdot)$ are plus-homogeneous maps. Therefore, map $g(\cdot)$ is type-K order-preserving and plus-homogeneous and so is the discrete-time system (5.8).

It is shown now that equilibrium points of vector field $f(\cdot)$ are fixed points of map $g(\cdot)$. Consider an equilibrium point x_e of $f(\cdot)$, i.e., $f(x_e) = 0$, then solutions $\varphi(t, x_e)$ starting at x_e remains in $\bar{x}$ for all $t \geq 0$, and thus x_e is a fixed point for map $g(\cdot)$.By Theorem 3.5.4 all periodic points of $g(\cdot)$ are fixed points, thus all periodic trajectories of $g(\cdot)$ are equilibrium points i.e.,

$$\lim_{k\to\infty} g^k(x) = \bar{x} \quad \Leftrightarrow \quad \lim_{t\to\infty} \varphi(t, x) = \bar{x}, \qquad \forall x \in \mathbb{R}^n$$

where $\bar{x}$ is a fixed point of $g(\cdot)$ or equivalently an equilibrium point of $f(\cdot)$. In other words, if one would assume the existence of a periodic trajectory of vector field $f(\cdot)$, this would imply a periodic point of map $g(\cdot)$, which contradicts Theorem 3.5.4. This completes the proof. □

In the next theorem, sufficient conditions on the structure of the local interaction rules of a discrete-time MAS as in (5.1) so that the vector field is type-K monotone and translation invariant, thus falling within the class of systems considered in Theorem 5.2.1 and ensuring the stability of the MAS for any initial condition in $\mathbb{R}^n$.

Theorem 5.2.2 *Consider n agents evolving according to*

$$\dot{x}_i(t) = f_i(x_i(t), x_j(t) : j \in \mathcal{N}_i).$$

If the MAS has at least one equilibrium point and if the set of differentiable local interaction rules $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, with $i = 1, \ldots, n$, satisfies the two conditions:

(i) $\partial f_i / \partial x_j \geq 0$ *for* $i \neq j$;
(ii) $f_i(x + \alpha \mathbb{1}) = f_i(x)$ *for any* $\alpha \in \mathbb{R}$;

then the MAS converges to one of its equilibrium points for any initial state $x(0) \in \mathbb{R}^n$.

Proof. If condition (i) is verified for any $i = 1, \ldots, n$, then the system is type-K monotone according to Proposition 5.1.1. We now show that if condition (ii) is verified for any $i = 1, \ldots, n$, then the system is translation invariant according to Definition 5.1.3. This can be easily proved by noticing that condition (ii) implies that the vector field $f(\cdot)$ is the same when computed at x and $x + \alpha \mathbb{1}$ for any $x \in \mathbb{R}^n$, thus, trajectories starting at these points are the same but shifted of $\alpha \mathbb{1}$, i.e.,

$$\varphi(t, x + \alpha \mathbb{1}) = \varphi(t, x) + \alpha \mathbb{1}, \quad \forall t \geq 0.$$

Since the system $f(\cdot)$ is a type-K monotone and translation invariant and it has at least one equilibrium point by the translation invariant property, one can exploits the result in Theorem 5.2.1 to establish that for any initial condition $x(0) \in \mathbb{R}^n$, the state trajectories of the MAS converge to one of its equilibrium points. □

As a special case, the consensus problem is studied. Two additional sufficient conditions are given. The first condition ensures that the consensus state is an equilibrium manifold for the MAS. The second condition is based on the inference graph $\mathcal{G}(f)$; it requires that there must exists a globally reachable node in graph $\mathcal{G}(f)$ and it implies that the consensus state manifold becomes globally asymptotically stable for the MAS. These two conditions applied to MASs as in eq. 4.1 satisfying conditions of Theorem 5.2.2 guarantee that the MAS asymptotically reaches the consensus state, as shown in the next theorem.

Theorem 5.2.3 *Consider n agents evolving according to*

$$\dot{x}_i(t) = f_i(x_i(t), x_j(t) : j \in \mathcal{N}_i).$$

If the set of differentiable local interaction rules $f_i : \mathbb{R}^n \to \mathbb{R}$, with $i = 1, \dots, n$, satisfies the next conditions:

(i) $\partial f_i / \partial x_j \geq 0$ for $i \neq j$;
(ii) $f_i(x + \alpha \mathbb{1}) = f_i(x)$ for any $\alpha \in \mathbb{R}$;
(iii) $f_i(\mathbb{0}) = 0$;
(iv) Inference graph $\mathcal{G}(f)$ has a globally reachable node;

then, the MAS converges asymptotically to a consensus state for any initial state $x(0) \in \mathbb{R}^n$.

Proof. Following the same reasoning of the proof of Theorem 5.2.2 [154], one can study the trajectories of the continuous-time system by equivalently studying the trajectories of the discrete-time system

[154] Mickens (1994), *Nonstandard finite difference models of differential equations.*

$$\hat{x}(k+1) = g(x(k)) = \varphi(T, \hat{x}(k)), \quad \forall x \in \mathbb{R}^n, \qquad (5.9)$$

for any $T \in \mathbb{R}$, in fact

$$\lim_{k \to \infty} g^k(x) \equiv \lim_{t \to \infty} \varphi(t, x), \quad \forall x \in \mathbb{R}.$$

Conditions (i) and (ii) imply that the continuous-time system is type-K monotone and translation invariant. Thus, due to type-K monotonicity, solutions $\varphi(\cdot)$ are type-K order-preserving maps. Due to translation invariance, solutions $\varphi(\cdot)$ are plus-homogeneous maps. Therefore, map $g(\cdot)$ is type-K order-preserving and plus-homogeneous and so is the discrete-time system (5.16). Furthermore, it is straightforward to notice that condition (iii) implies that $f(\mathbb{0}) = \mathbb{0}$ and condition (iv) implies that $\mathcal{G}(f)$ contains a globally reachable nodes.

Let us denote the consensus space as $\mathcal{C} = \{\alpha \mathbb{1} : \alpha \in \mathbb{R}\}$. Since all conditions of Theorem 4.2.6 are satisfied, it follows that for all $x \in \mathbb{R}^n$ it holds

$$\lim_{k \to \infty} g^k(x) \in \mathcal{C},$$

and thus the MAS converges asymptotically to a consensus state, completing the proof. □

Two general families

In this section two families of nonlinear control protocols are considered,

$$\dot{x}_i(t) = f_i(x(t)) = h_i\left(\sum_{j\in\mathcal{N}_i}(x_j(t) - x_i(t))\right), \tag{5.10}$$

$$\dot{x}_i(t) = f_i(x(t)) = \sum_{j\in\mathcal{N}_i} h_i\left(x_j(t) - x_i(t)\right), \tag{5.11}$$

where $h_i : \mathbb{R} \longrightarrow \mathbb{R}$. It is clear that if $h_i(x) = x$ for all i, both laws define the classical linear consensus protocol. Thus, the linear protocol is a special case of these two families. The key feature of these two families is that they satisfy the property of translation invariance by construction.

Theorem 5.2.4 *Consider n agents evolving according to protocol* (5.10) *or protocol* (5.11). *If the MAS has at least one equilibrium point and if the set of differentiable local interaction rules $h_i : \mathbb{R} \longrightarrow \mathbb{R}$, with $i = 1, \ldots, n$, satisfies the next condition:*

(i) $dh_i/\partial x \geq 0$ for $i \neq j$;

then the MAS converges to one of its equilibrium points for any initial state $x(0) \in \mathbb{R}^n$.

Proof. The proof is a straightforward consequence of Theorem 5.2.2. □

Theorem 5.2.5 *Consider n agents evolving in $\mathbb{R}$ according to protocol* (5.10) *or protocol* (5.11). *If the set of differentiable local interaction rules $h_i : \mathbb{R} \longrightarrow \mathbb{R}$, with $i = 1, \ldots, n$, satisfies the next conditions:*

(i) $\partial h_i/\partial x \geq 0$ for $i \neq j$;
(ii) $h_i(0) = 0$;
(iii) Inference graph $\mathcal{G}(f)$ has a globally reachable node;

then, the MAS converges asymptotically to a consensus state for any initial state $x(0) \in \mathbb{R}^n$.

Proof. The proof is a straightforward consequence of Theorem 5.2.3. □

These theorems extend results in [162, 237] to more general non-

[162] Munz et al. (2011), 'Consensus in multi-agent systems with coupling delays and switching topology'.
[237] Xu and Tian (2013), 'Design of a class of nonlinear consensus protocols for multi-agent systems'.

linear functions $h_i(\cdot)$. In particular, in [162] the family in (5.11) is considered but the nonlinear coupling $h_i(\cdot)$ is assumed to satisfy a passivity condition. On the other hand, in [237] the family in (5.10) is considered but the nonlinear coupling $f_i(\cdot)$ is assumed to be strictly increasing and odd.

[162] Munz et al. (2011), 'Consensus in multi-agent systems with coupling delays and switching topology'.

[237] Xu and Tian (2013), 'Design of a class of nonlinear consensus protocols for multi-agent systems'.

Both protocols can find their applications. Protocol in (5.11) can be utilized to analyze the synchronization of phase oscillators, while protocol in (5.10) can accommodate the saturation of control input by letting $f_i(\cdot) = \text{sat}(\cdot)$ regardless of the number of agents. These two are indeed the main applications presented in next sections.

5.3 Application to consensus with bounded control input

The most common consensus algorithms for continuous-time MASs $\dot{x}_i(t) = u_i$ is given by the following control input

$$u_i = \sum_{j \in \mathcal{N}_i} \left(x_j - x_i \right).$$

The problem to be studied here is to design proper saturating functions such that the above consensus protocol is yet qualifiable, when the inputs of all agents are subject to a saturation constraint

$$-s_i \leq u_i \leq s_i$$

where $s_i \in \mathbb{R}_+$ is a known positive scalar representing the constraint on the input amplitude. We consider the following generic saturating function $\text{sat}_i : \mathbb{R} \longrightarrow [-s_i, s_i]$,

$$\text{sat}_i(x) = s_i \left(\frac{1 - e^{-mx}}{1 + e^{-mx}} \right), \tag{5.12}$$

which encompasses several well-known saturating function, notably:

- $\text{sat}_i(x) = \tanh(x)$ if $s_i = 1$ and $m = 2$;
- $\text{sat}_i(x) = \text{sign}(x)$ if $s_i = 1$ and $m \longrightarrow \infty$;

It can be verified that a MAS wherein the agents are subject to one of the following saturated control action

$$u_i = \sum_{j \in \mathcal{N}_i} \text{sat}_i \left(x_j(t) - x_i(t) \right),$$
$$u_i = \text{sat}_i \left(\sum_{j \in \mathcal{N}_i} \left(x_j(t) - x_i(t) \right) \right). \tag{5.13}$$

achieve consensus if the underlying graph contains a globally reachable node. This is due to Theorem 5.2.5, in fact: conditions (i) and (ii) hold due to the following properties of the saturating function,

$$\text{sat}_i(0) = 0, \qquad \frac{\partial}{\partial x}\text{sat}_i \geq 0,$$

and condition (iii) is satisfied by assumption.

In the following we provide some numerical simulations in a network of 5 nodes with a directed line configuration. We consider the case in which saturation is applied to the sum of the inputs with saturation bound $s_i = 1$ for all agents. The dynamics of the considered network takes the form

$$\dot{x}(t) = -\text{sat}(Lx(t)),$$

and Figure 5.1 shows some simulations for $t = \varepsilon k$. In the top figure, the evolution of the system without the saturation of the inputs is provided. Then, in the middle figure, the saturation is applied with $m = 2$, thus resulting in the hyperbolic tangent $\text{sat}(x) = \tanh(x)$. Notice that the the saturation of the inputs entails a slowdown of the convergence rate. Finally, in the bottom figure the saturation is applied with an high value of $m = 1000$, thus well approximating the sign function $\text{sat}(x) \approx \text{sign}(x)$. It can be founded that, owing to the saturation, the state of the agents starts varying with a constant rate $s_i = 1$, instead of an exponential rate in the case without saturation, which is recovered when the agents get close enough and the saturation does not take place anymore. Further, it can be concluded that when $m \to \infty$ the consensus will be achieved in finite time and the control law can be regarded as a discontinuous control law.

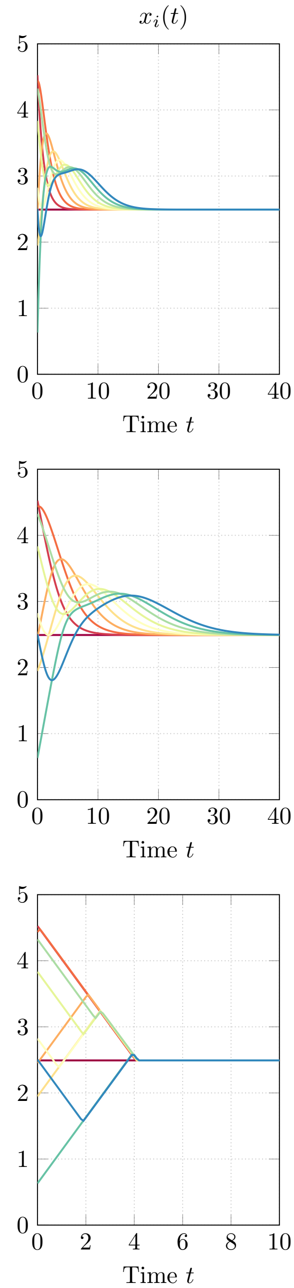

Figure 5.1: Evolution of a network of 5 agents in a directed line configuration. In the top figure the input is not saturated; in the middle and bottom figure the input is saturated according to (5.12) with $m = 2$ and $m = 1000$, respectively.

5.4 Application to synchronization of phase-coupled oscillators

The emergence of synchronization in a network of coupled oscillators is a fascinating subject of multidisciplinary research.Within the rich modeling phenomenology on synchronization among coupled oscillators, this section focuses on diffusively coupled phase oscillators. A single uncoupled phase oscillator is characterized by a phase $\theta_i(t)$ that evolves on the circle with constant velocity $\dot{\theta}_i(t) = \omega_i$, where ω_i is the natural frequency of the oscillator. Within the network, n phase oscillators are coupled through their phase differences according to a graph $\mathcal{G}$ and coupling functions h_{ij}. The dynamics of each oscillator is given by

$$\dot{\theta}_i(t) = \omega_i + \sum_{j \in \mathcal{N}_i} h_{ij}(\theta_j(t) - \theta_i(t)). \tag{5.14}$$

Weakly-coupled identical limit-cycle oscillators can be well approximated by this canonical model through a phase reduction and averaging analysis, with appropriate coupling functions h_{ij} that are closely related to the phase response curve of the oscillators. Since the phase response curve is a function computed on the periodic limit cycle, it is 2π-periodic and so are the coupling functions h_{ij}.

We provide a new analysis tool for studying synchronization in networks of identical oscillators, $\omega_i = \omega$, with directed and heterogeneous couplings satisfying the following condition,

$$\frac{d}{d\theta} h_{ij}(\theta) = \begin{cases} > 0 & \theta \in (-\alpha, \alpha) \\ < 0 & \theta \in (-\pi, -\alpha, \alpha, \pi) \end{cases}, \tag{5.15}$$

with $\alpha \in [0, \pi]$ and $h_{ij}(0) = 0$: Figure 5.2 provides a graphical representation of this condition.

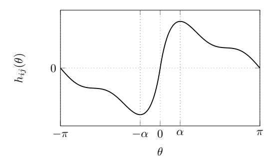

Figure 5.2: The 2π-periodic coupling function satisfying the piecewise monotonicity.

Let a, b be any real numbers such that $0 \leq b - a \leq 2\alpha$ and define $D = [a, b]^n \subset \mathbb{R}^n$. It can be checked that D is an invariant space for the system and all conditions of Theorem 5.2.3 are satisfied if the graph is assumed to contain a globally reachable node. Therefore, for any initial condition $\theta(0) \in D$ the oscillators reach consensus, i.e., phase synchronization.

Moreover, the following theorem shows that it is always possible to design α such that the phase synchronized state is the unique stable configuration of the network, thus providing global synchronization for initial configuration in the whole torus. In fact, by condition in eq. (5.15) it always holds that the synchronized state is stable, and moreover α can always be chosen smaller enough to make unstable all other equilibrium configurations.

Theorem 5.4.1 *Consider a network of identical oscillators with dynamics as in eq.* (5.14). *A configuration θ^* which constitutes an equilibrium of the network is:*

- *stable if the Jacobian $J_f(\theta^*)$ is Metzler;*
- *unstable if the negative Jacobian $-J_f(\theta^*)$ is Metzler.*

Proof. The proof is given in the next page. □

The considered class of coupling functions encompasses several oscillator models, for instance:

- Kuramoto oscillators with $h_{ij}(\theta) = \sin(\theta)$ with $\alpha = \pi/2$ [128].
- Monotone oscillators with $\alpha \in \{0, \pi\}$ and any strictly monotone function $h(\theta)$ such that $h_{ij}(\theta) = h(\theta)$ [150];
- Oscillators with symmetric and odd coupling [146, 147].

The synchronization results provided in this section, constitutes an alternative proof to the result provided in [253] specialized for Kuramoto oscillators coupled according to a fixed directed graph with a globally reachable node and initial state confined in the semicircle. Moreover, they provide an extension of the result provided in [150] for monotone oscillators to the case of a non-complete and directed graph with initial state in the whole circle.

As an example, we provide in Figure 5.3 simulations of a network of 10 oscillators coupled according to a directed circle graph. The network has two phase configuration which constitute an equilibrium, which are characterized by all phase differences $|\theta_{i+1} - \theta_i|$ being either equal to zero (i.e., the phase-synchronized state θ_{syn}) or equal to $\bar{\alpha} = \pi/5$ (i.e., the splay state θ_{splay}). In the top figure, Kuramoto oscillators are considered, for which it holds $\alpha = \pi/2$. Since $\alpha > \bar{\alpha}$, the Jacobian matrix at the splay state θ_{splay} is Metzler and therefore it is a stable configuration. In fact, oscillators starting close enough to this undesired equilibrium, converge to it. In the middle figure, generic coupling functions satisfying condition (5.15) is considered, with $\alpha = \pi/6$. Since $\alpha < \bar{\alpha}$, the

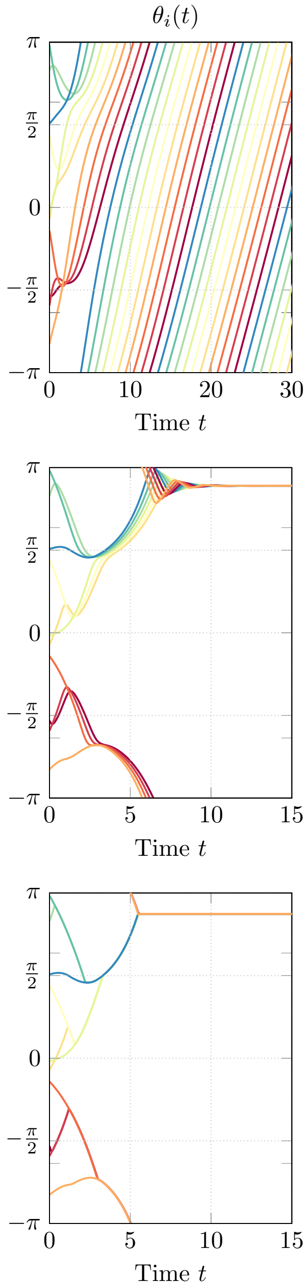

Figure 5.3: Evolution of a network of 10 oscillators in a directed circle configuration. The coupling function are designed with $\alpha = \pi/2$ (top figure), $\alpha = \pi/6$ (middle figure) and $\alpha \approx 0$ (bottom figure).

[128] Kuramoto (1975), 'Self-entrainment of a population of coupled nonlinear oscillators'.

negative Jacobian matrix at the splay state θ_{splay} is Metzler and thus it is an unstable configuration. The only equilibrium point is the synchronized state, and the network globally converge to it for any initial configuration of the oscillators. In the bottom figure, "almost" monotone oscillators are considered, i.e., the coupling function common to all oscillator satisfying condition (5.15) is selected with $\alpha \approx 0$ but $\alpha > 0$. We highlight that the smaller is the value of α, and the higher is the rate of convergence to the synchronized state. In particular, as $\alpha \to 0$, synchronization is reached in finite-time; this result is coherent with the results provided in [150].

[150] Mauroy and Sepulchre (2012), 'Contraction of monotone phase-coupled oscillators'.

[147] Mallada and Tang (2010), 'Synchronization of phase-coupled oscillators with arbitrary topology'.

[146] Mallada et al. (2015), 'Distributed synchronization of heterogeneous oscillators on networks with arbitrary topology'.

[253] Zhiyun et al. (2007), 'State Agreement for Continuous-Time Coupled Nonlinear Systems'.

Proof of Theorem 5.4.1

Following the same reasoning of the proof of Theorem 5.2.2 [154], one can study the trajectories of the continuous-time system by equivalently studying the trajectories of the discrete-time system

$$\hat{\theta}(k+1) = g(\theta(k)) = \varphi(1, \hat{\theta}(k)), \quad \forall \theta \in \mathbb{R}^n, \tag{5.16}$$

in fact

$$\lim_{k\to\infty} g^k(\theta(0)) \equiv \lim_{t\to\infty} \varphi(t, \theta(0)), \quad \forall x \in \mathbb{R}.$$

[150] Mauroy and Sepulchre (2012), 'Contraction of monotone phase-coupled oscillators'.

[154] Mickens (1994), *Nonstandard finite difference models of differential equations*.

Let θ^* be an equilibrium point of the system, then it is a fixed point of the discrete mapping $g(\cdot)$ and by plus-homogeneity property the points in the manifold $\theta^* + c\mathbb{1}$ with $c \in \mathbb{R}$ are also fixed points. Consider a neighborhood W delimited by $a = \theta^* - c\mathbb{1}$ and $a = \theta^* + c\mathbb{1}$ with $\alpha > 0$, $\alpha \approx 0$, i.e., $W = [a, b]$, for which $a \leq \theta^* \leq b$. Then,

(i) if J_f is Mezler, then f is monotone in W by Lemma 5.1.1 and g is monotone in W by Lemma 3.2.1. Thus, for any $\theta \in W$, it follows

$$a = g^k(a) \leq g^k(\theta) \leq g^k(b) = b, \quad \forall k \geq 0.$$

Therefore, $g^k(W) \subset W$, and the equilibrium point is stable.

(ii) if $-J_f$ is Mezler, then by trivial generalization Lemmas 5.1.1-3.2.1, f is antitone and so is g, i.e., $x \leq y \Rightarrow g(x) \geq g(y)$ [134]. Thus, for any $\theta \in W$, it follows

$$a = g(a) \geq g(\theta), \quad \text{or} \quad g(\theta) \geq g(b) = b.$$

Therefore, $g(W) \not\subset W$, and the equilibrium point is unstable.

[134] Lemmens and Nussbaum (2012), *Nonlinear Perron-Frobenius Theory*.

Consensus for desynchronization and dynamic estimation

Desynchronization via Fiedler vector estimation 6

In this chapter we are mainly concerned with the problem of inducing desynchronization in a network of diffusively coupled harmonic oscillators with identical frequency $\omega \in \mathbb{R}$,

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t) \\ y_i(t) &= Cx_i(t) \end{aligned}, \qquad A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \quad B = C^\top = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

in a distributed way by designing a local control feedback. The basic idea underlying the design of the proposed local feedback is to exploit the zero mean property of the Fiedler vector of the Laplacian matrix encoding the diffusive coupling among oscillators in order to induce desynchronization in the network. It turns out that the proposed feedback constitutes a novel protocol to estimate the Fiedler vector in network of single integrators

$$\dot{y}_i(t) = u_i(t). \tag{6.1}$$

For the sake of clarity, after having reviewed the related literature in Section 6.1, we first present the protocol computing the Fiedler eigenvector in Section 6.2. Second, in Section 6.3 we employ a the proposed protocol as a local feedback law to desynchronize a network of coupled harmonic oscillators by driving it toward a state proportional to the Fiedler vector. Finally, in Section 6.4 numerical simulations corroborating the theoretical results are provided.

We highlight that only in this chapter, and only for the presentation of the Fiedler vector estimation protocol, we use $y(t)$ to denote the state of the agents instead of $x(t)$. This choice allows a coherent notation with the desynchronization protocol.

6.1 Related literature

Fiedler vector estimation

The computation of eigenvectors of the graph Laplacian L is a problem of fundamental importance for various applications and it is the cornerstone of spectral graph theory [38]. Among all eigenvectors, the Fiedler vector [74] plays a pivotal role: it is the eigenvector corresponding to the second smallest eigenvalue of the Laplacian matrix, also known as the algebraic connectivity. To

[38] Chung and Graham (1997), *Spectral graph theory*.

[74] Fiedler (1973), 'Algebraic connectivity of graphs'.

name a few, Fiedler vector is useful in graph partitioning [25, 97, 214] and in the control of algebraic connectivity [52, 241, 245].

Power Iteration (PI) [220] is an established iterative method to compute the leading eigenvalue(s) and associated eigenvector(s) of a matrix. In [26, 52, 63, 241, 244] the Fiedler vector is computed by means of methods based on a distributed implementation of PI. Main drawbacks of [52], which exploits the algorithm proposed in [115], are the centralized initialization step and the high number and size of the messages the nodes need to exchange. In [241] and [63] the decentralization is carried on at each agent by two consensus estimators, which are required to run "fast enough" in order to expect the resulting dynamics to converge: a formal proof is not provided. Similar approaches are used to compute eigenvalues and the algebraic connectivity [178]. Another algorithm with application to topology inference in ad hoc network has been proposed in [25, 26]. Another class of algorithms forces the nodes to oscillate at eigenfrequencies and deduce spectral information through Fast Forurier Transform (FFT). In [198] the Fiedler vector is computed by running at every node the wave equation and computing the eigenvector components through an FFT. This algorithm is proved to be orders of magnitude faster than PI-based algorithms. An FFT approach for distributedly computing the eigenvalues is given in [83]. On one hand, the main limitation of PI-based approaches consists on the distributed normalization of the vectors at each step, which severely affects their convergence speed and requires a centralized initialization step. On the other hand, FFT-based approaches suffer from a rather poor accuracy and robustness issues.

The protocol proposed in Section 6.2 relies neither on a distributed PI nor a FFT approach, thus guaranteeing robustness to initial conditions, high convergence speed and accuracy. However, it requires the knowledge of the algebraic connectivity, which is a reasonable assumption for static networks (as in the case of our main application) since various distributed algorithms have been proposed to distributedly estimate all the eigenvalues of undirected graph Laplacian [83, 122, 219, 240].

[97] Hagen and Kahng (1992), 'New spectral methods for ratio cut partitioning and clustering'.
[214] Spielman and Teng (2007), 'Spectral partitioning works: Planar graphs and finite element meshes'.
[25] Bertrand and Moonen (2013), 'Seeing the bigger picture: How nodes can learn their place within a complex ad hoc network topology'.
[52] De Gennaro and Jadbabaie (2006), 'Decentralized control of connectivity for multi-agent systems'.
[245] Zavlanos et al. (2009), 'Hybrid control for connectivity preserving flocking'.
[241] Yang et al. (2010), 'Decentralized estimation and control of graph connectivity for mobile sensor networks'.
[220] Trefethen and Bau III (1997), *Numerical linear algebra*.

Desynchronization of harmonic oscillators

Due to their theoretical and practical significance, networks of harmonic oscillators have been experiencing growing attention and have been applied to address several problems electrical networks [223], in quantum electronics-mechanics-optics [28, 155, 247], resonance phenomena[208], motion coordination [21, 141] and acoustic vibrations [252]. One of the pioneering works studying synchronization in ensembles of harmonic oscillators is the one of W. Ren [192]. The convergence conditions over directed fixed networks are derived for harmonic oscillators with diffusive coupling. The construction of a dynamic output feedback coupling that achieves synchronization in uniformly connected networks is due to [203], while the case of proximity networks, in which the coupling depends on the relative distance between the oscillators, has been addressed in [216]. These results have been extended in [234] to deal with heterogeneous oscillators. Other topics of interest include finite-time through output feedback [67, 249], sampled-data coupling [231, 232, 246, 247], bipartite consensus [140, 212].

While synchronization has been formally defined and thorougly investigated for harmonic oscillators, a definition of desynchronization is missing. Even for first order nonlinear oscillators, such as Kuramoto oscillators, the definition of desynchronization is ambiguous as it depends upon the metric used to measure the degree of synchronization of a network and the underlying mathematical model of the network system. Desynchronization has significance in several fields. Some neuropathologies (e.g., epilepsy and Parkinson) are believed to be related to abnormal synchronization of neurons; treatment of these neurological diseases is thus addressed by means of desynchronization, and different oscillator models, such as Kuramoto oscillators [84, 156, 157], impulse-coupled oscillators [187], Stuart-Landau oscillators [158, 180], FitzHugh-Nagumo oscillators [2, 116, 189], Andronov-Hopf oscillators [181]. Desynchronization is a useful primitive for periodic resource sharing and applies to sensor network applications: it implies that nodes perfectly interleave periodic events thus allowing to evenly distribute sampling burden in a group of nodes, schedule sleep cycles, or organize a collision-free time division multiple access schedule for transmitting wireless messages [44, 54, 153, 184]. Another important problem is the motion coordination [15, 34, 90, 135].

The desynchronization measure proposed in Section 6.3 is the first formulated for harmonic oscillators. On the basis of this definition, we propose a local control protocol for the oscillators to desynchronize in a connected undirected network.

[52] De Gennaro and Jadbabaie (2006), 'Decentralized control of connectivity for multi-agent systems'.

[241] Yang et al. (2010), 'Decentralized estimation and control of graph connectivity for mobile sensor networks'.

[26] Bertrand and Moonen (2013), 'Distributed computation of the Fiedler vector with application to topology inference in ad hoc networks'.

[63] Di Lorenzo and Barbarossa (2014), 'Distributed estimation and control of algebraic connectivity over random graphs'.

[244] Zareh et al. (2018), 'Distributed laplacian eigenvalue and eigenvector estimation in multi-robot systems'.

[178] Oreshkin et al. (2010), 'Optimization and analysis of distributed averaging with short node memory'.

[198] Sahai et al. (2010), 'Wave equation based algorithm for distributed eigenvector computation'.

[83] Franceschelli et al. (2013), 'Decentralized estimation of Laplacian eigenvalues in multi-agent systems'.

[192] Ren (2008), 'Synchronization of coupled harmonic oscillators with local interaction'.

[203] Scardovi and Sepulchre (2009), 'Synchronization in networks of identical linear systems'.

6.2 Proposed distributed protocol to estimate the Fiedler vector

Consider a network of n single integrators

$$\dot{y}_i(t) = u_i(t), \qquad \forall i = 1, \dots, n, \tag{6.2}$$

where $y_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are, respectively, the state and the input of agent i. The network's topology is represented by an undirected graph $\mathcal{G} = (V, E)$ and the control law u_i is said to be local if each agent can only exchange information with its neighbors. The goal of this section to design a local control law such that each agent i estimates the i-th Fiedler vector component of the Laplacian matrix L of $\mathcal{G}$. For the sake of thoroughness, we first provide a global control law from which a local version is derived in the next subsection.

[216] Su et al. (2009), 'Synchronization of coupled harmonic oscillators in a dynamic proximity network'.

[234] Wieland et al. (2011), 'An internal model principle is necessary and sufficient for linear output synchronization'.

Theorem 6.2.1 *Consider a MAS with agents dynamics* (6.2) *driven by the control law*

$$u_i(t) = \sum_{j \in \mathcal{N}_i} \big(y_j(t) - y_i(t)\big) - \alpha\, \mathbb{1}^\top y(t) + \lambda_{L,2} y_i(t). \tag{6.3}$$

If $\mathcal{G}$ is connected and $\alpha > \lambda_{L,2}/n$ then the MAS converges to a state proportional to the Fiedler vector of graph $\mathcal{G}$.

Proof. The dynamics can be written in compact form as

$$\dot{y}(t) = -\underbrace{[L + \alpha \mathbb{1}\mathbb{1}^\top - \lambda_{L,2} I]}_{M}\, y(t),$$

Where L denotes the Laplacian matrix as in Definition 2.2.1. For an undirected graph $\mathcal{G}$ without self-loops it is easy to derive that L is a symmetric and positive semidefinite, i.e., all eigenvalues of L are real and non-negative. In this case, by convention, we write these eigenvalues as

$$0 = \lambda_{L,1} \le \lambda_{L,2} \le \cdots \le \lambda_{L,n}.$$

To the eigenvalue $\lambda_{L,1} = 0$ is associated the eigenvector $v_{L,1} = \mathbb{1}$ and to the eigenvalue $\lambda_{L,2}$, which is known as the *algebraic connectivity*, is associated the eigenvector $v_{L,2}$, which is known as the *Fiedler vector*. The construction of matrix M consists of two conceptual steps.

1. *Matrix inflation*: add a term $\alpha\mathbb{1}\mathbb{1}^\intercal$ to the Laplacian matrix L, obtaining $N = L + \alpha\mathbb{1}\mathbb{1}^\intercal$. Note that matrix N has the following properties:
   - ▶ $\mathbb{1}$ is an eigenvector of N associated to the eigenvalue αn. To prove this, we have:

     $$N\mathbb{1} = (L + \alpha\mathbb{1}\mathbb{1}^\intercal)\mathbb{1} = \cancel{L\mathbb{1}} + \alpha\mathbb{1}\mathbb{1}^\intercal = \alpha n\mathbb{1}.$$

   - ▶ Any other eigenvector v^* of L with associated eigenvalue λ^* is an eigenvector of N with the same associated eigenvalue λ^*. To prove this, we have:

     $$Nv^* = (L + \alpha\mathbb{1}\mathbb{1}^\intercal)v^* = Lv^* + \cancel{\alpha\mathbb{1}\mathbb{1}^\intercal v^*} = \lambda^* v^*,$$

     where the middle step follows from the fact that the eigenvectors of a symmetric matrix are orthogonal.

   It is straightforward to conclude that the smallest eigenvalue of matrix N is $\lambda_{N,1} = \lambda_{L,2}$ if $\alpha > \lambda_{L,2}/n$ and its associated eigenvector is the Fiedler vector $\lambda_{N,1} = \lambda_{L,2}$.
2. *Eigenvalues shifting*: subtract matrix $\lambda_{L,2}I$ to the resulting matrix, obtaining $M = N - \lambda_{L,2}I$. All eigenvalues are shifted by $\lambda_{L,2}$ thus ensuring that $\lambda_{M,1} = 0$ is a single null eigenvalue with associated eigenvector $\lambda_{N,1} = \lambda_{L,2}$.

It follows that matrix M has spectrum

$$0 = \lambda_{M,1} < \lambda_{M,2} \leq \cdots \leq \lambda_{M,n}$$

with $v_{M,1} = v_{L,2}$ and $\lambda_{M,1} = \lambda_{L,2}$. Thus, the MAS is marginally stable and y converges asymptotically to the Fiedler vector, completing the proof. □

The control law (6.3) is not local as it relies on two global information: the algebraic connectivity λ_2 and an average of the state variables $\mathbb{1}^\intercal y(t)$. In this work we make the standing assumption that the algebraic connectivity λ_2 is known. Such an assumption is plausible for static networks since it can be inferred in a distributed way by several algorithms [83, 122, 219, 240]. On the other hand, the problem of computing in real time the average $\mathbb{1}^\intercal y(t)$ can be overcome by employing a distributed estimator of this quantity, as proposed in earlier work [85, 120, 202]. In particular, we consider an integral dynamic consensus algorithm, which differs from those presented in [120].

[122] Kibangou and Commault (2012), 'Decentralized Laplacian eigenvalues estimation and collaborative network topology identification'.
[83] Franceschelli et al. (2013), 'Decentralized estimation of Laplacian eigenvalues in multi-agent systems'.
[219] Tran and Kibangou (2015), 'Distributed estimation of Laplacian eigenvalues via constrained consensus optimization problems'.
[240] Yang and Tang (2015), 'Distributed estimation of graph spectrum'.
[85] Freeman et al. (2006), 'Stability and convergence properties of dynamic average consensus estimators'.
[202] Scardovi and Sepulchre (2006), 'Collective optimization over average quantities'.
[120] Kia et al. (2019), 'Tutorial on Dynamic Average Consensus: The Problem, Its Applications, and the Algorithms'.
[120] Kia et al. (2019), 'Tutorial on Dynamic Average Consensus: The Problem, Its Applications, and the Algorithms'.

Theorem 6.2.2 *Consider a MAS with agents dynamics* (6.2) *driven by the control law*

$$u_i(t) = \sum_{j\in\mathcal{N}_i} \left(y_j(t) - y_i(t)\right) - \alpha v_i(t) + \lambda_{L,2} y_i(t). \tag{6.4}$$

where $v_i(t) \in \mathbb{R}$ *is given by*

$$\begin{aligned} \dot{v}_i(t) &= \dot{y}_i(t) + \beta(y_i(t) - v_i(t)) + K_I \sum_{j\in\mathcal{N}_i} \left(z_j(t) - z_i(t)\right), \\ \dot{z}_i(t) &= K_I v_i(t), \end{aligned} \tag{6.5}$$

and $\alpha, \beta, K_I \in \mathbb{R}$ *are constants. If* $\mathcal{G}$ *is connected and if*

$$\alpha > \lambda_{L,2}, \quad \beta > 0, \quad K_I > 0,$$

then $y(t)$ *converges almost globally to a state proportional to the Fiedler vector of graph* $\mathcal{G}$.

Proof. Let $w(t) = \begin{bmatrix} y(t) & v(t) & z(t) \end{bmatrix}^\top \in \mathbb{R}^{3n}$ denote the augmented state of the network. Denoting with $I = I_n$ the identity matrix of dimension n, the dynamics can be written in compact form as

$$\dot{w}(t) = \underbrace{\begin{bmatrix} \lambda_{L,2} I - L & -\alpha I & \mathbb{0} \\ (\lambda_{L,2} + \beta) I - L & -(\alpha + \beta) I & -K_I L \\ \mathbb{0} & K_I I & \mathbb{0} \end{bmatrix}}_{M} w(t).$$

The streamline of the proof is as follows:

1. We derive a closed form for the eigenvalues of matrix $M \in \mathbb{R}^{3n\times 3n}$ as a function of the eigenvalues $\lambda_{L,i}$ of the Laplacian matrix $L \in \mathbb{R}^{n\times n}$;
2. We prove that M has a zero eigenvalue with algebraic and geometric multiplicity equal to 2;
3. By means of the Routh criteria, we show that all other eigenvalues of M have strictly negative real part;
4. Finally, we show that the space spanned by the two distinct eigenvectors associated to the zero eigenvalue coincides with the space spanned by the Fiedler vector for the component $y(t)$ of $w(t)$, thus completing the proof.

In the following, these steps are identified with a margin note for the sake of clarity in the presentation.

Step 1: Derivation of a closed form for the eigenvalues of M.

To compute the eigenvalues of matrix M, we proceed by solving

$\det\{M - \lambda I_{3n}\} = 0$. Let us partition matrix $M - \lambda I_{3n} = [A\ \ B; C\ \ D]$ where

$$A = \begin{bmatrix} (\lambda_{L,2} - \lambda)I - L & -\alpha I \\ (\lambda_{L,2} + \beta)I - L & -(\alpha + \beta + \lambda)I \end{bmatrix}, B = \begin{bmatrix} \mathbb{0} \\ -K_I L \end{bmatrix},$$
$$C = \begin{bmatrix} \mathbb{0} & K_I I \end{bmatrix}, \quad D = -\lambda I.$$

By Shur-complement, the determinant of $M - \lambda I_{3n}$ is equal to $\det\{D\} \cdot \det\{A - BD^{-1}C\}$, where $\det\{D\} = (-\lambda)^n$ and $\det\{A - BD^{-1}C\} = \det\{a_1 L^2 + a_2 L + a_3 I\}$, with

$$a_1(\lambda) = \frac{K_I^2}{\lambda}, \qquad a_2(\lambda) = K_I^2 - \lambda_{L,2}\frac{K_I^2}{\lambda} + \beta + \lambda,$$
$$a_3(\lambda) = \lambda\ \alpha + \lambda\ \beta + \lambda^2 - \lambda_{L,2}\ \beta - \lambda_{L,2}\ \lambda + \alpha\beta.$$

Finally, letting $H(\lambda) = a_1(\lambda)L + a_2(\lambda)I$, we can write

$$\det\{M - \lambda I_{3n}\} = (-\lambda)^n \cdot \det\{H(\lambda)L + a_3(\lambda)I\} = 0. \qquad (6.6)$$

We note that:

- ▶ The eigenvalues of $H(\lambda)$ are denoted $\lambda_{H,i}(\lambda)$ and are given by $\lambda_{H,i} = a_1(\lambda)\ \lambda_{L,i} + a_2(\lambda)$.
- ▶ Matrix $H(\lambda)L$ is a product of two commuting matrices, i.e., $H(\lambda)L = LH(\lambda)$, thus any eigenvalue of $H(\lambda)L$ is a product of the eigenvalues of $H(\lambda)$ and L.

In the light of the above considerations, from eq. (6.6) we derive the next relationships

$$\lambda(\lambda_{L,i}\lambda_{H,i}(\lambda) + a_3(\lambda)) = 0, \qquad \forall i \in V,$$

and thus by substitution $\forall i \in V$ we obtain

$$\lambda^3 + b_i\lambda^2 + c_i\lambda + d_i = 0, \quad \text{with} \quad \begin{aligned} b_i &= \alpha + \beta - \lambda_{L,2} + \lambda_{L,i} \\ c_i &= (\alpha - \lambda_{L,2})\beta + (K_I^2 + \beta)\lambda_{L,i}\ . \\ d_i &= K_I^2 \lambda_{L,i}(\lambda_{L,i} - \lambda_{L,2}) \end{aligned}$$

One can notice that M has three eigenvalues for each $\lambda_{L,i}$ with $i \in V$, which can be computed by the above equation. For $i = \{1, 2\}$ it holds $d_i = 0$ and $\lambda = 0$ is a solution; moreover, by the Routh criteria specialized for second degree polynomials, all other solutions are strictly negative if and only if coefficients b_i, c_i are positive, which is verified if the conditions of the theorem hold.

Step 2: M is shown to have a zero eigenvalue with algebraic multiplicity equal to 2.

For $i \in \{3, \ldots, n\}$ it holds $d_i > 0$ and, by the Routh criteria specialized for third degree polynomials, all solutions are strictly negative

Step 3: all other eigenvalues are shown to have strictly negative real part.

if and only if in addition it holds $b_i c_i - d_i > 0$, proved next

$$\begin{aligned} b_i c_i - d_i &= \beta(\lambda_{L,2} - \lambda_{L,i})^2 + (2\alpha\beta + \beta^2)(\lambda_{L,i} - \lambda_{L,2}) + \\ &+ (\alpha\beta + K_I^2 \lambda_{L,i})(\alpha + \beta) > 0, \end{aligned}$$

since $\lambda_{L,i} \geq \lambda_{L,2}$ for $i \in \{3, \dots, n\}$.

Step 4: Two distinct eigenvectors are associated to the zero eigenvalue.

We conclude that, under the conditions of the theorem, matrix M has two zero eigenvalues because $d_i = 0$ for $i \in \{1, 2\}$ while all other eigenvalues have negative real part. Stability of the system can be ensured by proving that the geometric multiplicity of eigenvalue 0 is equal to its algebraic multiplicity, which is two. We prove this fact by showing that two distinct eigenvectors are associated to the zero eigenvalue. Recalling that $w(t) = \begin{bmatrix} y(t) & v(t) & z(t) \end{bmatrix}^\top \in \mathbb{R}^{3n}$ is the state of the overall system, we compute $\dot{w}(t) = Mw(t) = \mathbb{0}$,

$$\begin{cases} (\lambda_{L,2} I - L) y(t) = \mathbb{0} \\ \beta y(t) - K_I L z(t) = \mathbb{0} \\ v(t) = \mathbb{0} \end{cases},$$

There are two feasible choices for y. First choice is $y(t) = \mathbb{0}$, leading to $Lz(t) = \mathbb{0}$, i.e., $z(t) = \delta v_{L,2}, \forall\, \delta \in \mathbb{R}$. Second choice is $y(t) = \delta v_{L,2}$, leading to $z(t) = \frac{\beta\delta}{K_I \lambda_{L,2}} v_{L,2} + \sigma\, \mathbb{1}, \forall\, \sigma \in \mathbb{R}$. Thus, the zero eigenvalue has the two distinct eigenvectors

$$e_1 = \begin{bmatrix} \mathbb{0} \\ \mathbb{0} \\ v_{L,2} \end{bmatrix}, \quad e_2 = \begin{bmatrix} v_{L,2} \\ \mathbb{0} \\ \frac{\beta}{K_I \lambda_{L,2}} v_{L,2} + \frac{\sigma}{\delta}\, \mathbb{1} \end{bmatrix}. \tag{6.7}$$

We conclude that the system is marginally stable, and

$$\lim_{t \to \infty} y(t) = \gamma v_{L,2}, \qquad \gamma \in \mathbb{R}.$$

Let $S \subset \mathbb{R}^{3n}$ be the space orthogonal to e_2. Coefficient γ is null if and only if $w(0) \in S$, which is a set of measure zero with respect to $\mathbb{R}^2$. Thus, if $w(0) \notin S$ then $y(t)$ almost globally converges to the Fiedler vector, completing the proof. □

Theorem 6.2.2 ensures that

$$\lim_{t \to \infty} y(t) = \gamma v_{L,2}$$

with $\gamma \neq 0$ if and only if the initial condition of the system is not orthogonal to the eigenvector e_2 given in (6.7). In the vicinity of this critical hyperplane of initial conditions, γ can be small thus leading to a large numerical error in the estimation of the Fiedler vector. However, we point out that this critical set has measure zero with respect to $\mathbb{R}^2$. Moreover, if a minimal amount o noise affecting the state variables is considered, then the condition of $\gamma \neq 0$ is always satisfied.

6.3 Application to desynchronization of coupled harmonic oscillators

A harmonic oscillator is a second-order linear system modeling both amplitude $M(t)$ and phase $\theta(t)$ of an oscillator. An harmonic oscillator i with natural frequency ω has dynamics given by

$$\dot{x}_i(t) = \underbrace{\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}}_{A} x_i(t). \tag{6.8}$$

Consider a network of n identical harmonic oscillators

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t), \qquad i \in V, \\ y_i(t) &= Cx_i(t), \end{aligned} \tag{6.9}$$

where $x_i \in \mathbb{R}^2$ is the state of the oscillator, $u_i \in \mathbb{R}$ is the control input, $y_i \in \mathbb{R}$ is the output, matrix A is given in (6.8), matrices B and C are $C = B^\intercal = [0 \quad 1]$. The input

$$u_i = u_i^d + u_i^c \in \mathbb{R}$$

consists of two terms:

- the diffusive coupling $u_i^d \in \mathbb{R}$ among the oscillators, which is given by

$$u_i^d(t) = \sum_{j \in \mathcal{N}_i} \left(y_j(t) - y_i(t) \right), \quad i \in V; \tag{6.10}$$

- The local control feedback $u_i^c \in \mathbb{R}$ to be designed.

We assume that the coupling network and the communication network coincide and thus they can be both modeled by the same graph $\mathcal{G}$: $j \in \mathcal{N}_i$ if and only if oscillator i is coupled and can communicate with oscillator j.

Desynchronization measure

In this section we define and study the desynchronization problem in networks of n coupled identical harmonic oscillators with natural frequency $\omega \in \mathbb{R}$.

Kuramoto [128, 129] considers oscillators described by phasors, i.e., vectors rotating about the origin in the complex plane. When all phasors have the same amplitude, which is assumed unitary, their end points move along the unit circle. Kuramoto used the magnitude of the centroid of these points as a synchronization measure: a network of oscillators is said to be synchronized if the centroid is on the circle, i.e., its magnitude is one.

[128] Kuramoto (1975), 'Self-entrainment of a population of coupled nonlinear oscillators'.
[129] Kuramoto (2003), *Chemical oscillations, waves, and turbulence*.

We consider a more general case, where harmonic oscillators are described by vectors of possibly different magnitude, rotating about the origin of the complex place. Also in this case one can compute the centroid of the end points of these vectors. We say that a network of oscillators is desynchronized if the centroid is at the origin of the complex plane, i.e., its magnitude is null.

In the case of Kuramoto oscillators, the magnitude of the centroid is is usually referred in the literature as the *order parameter* [129]; the concept of order parameter is useful since it characterizes the average dynamical behavior of the system. In the following, we provide a generalization of the definition of order parameter to networks of oscillators with different amplitude.

[129] Kuramoto (2003), *Chemical oscillations, waves, and turbulence*.

Let the steady state output of each oscillator i be given by

$$y_i^{ss}(t) = M_i \cos(\omega t + \theta_i) = \Re\left\{M_i e^{j\theta_i} \cdot e^{j\omega t}\right\},$$

where $M_i \in \mathbb{R}_+$, $\theta_i \in \mathbb{S}^1$ with S^1 denoting the unit circle, are the steady state magnitude, respectively, and the phase of oscillator i, j denotes the imaginary unit and $\Re\{\cdot\}$ denotes the real part of a complex number. Thus, the collective steady-state output dynamics

$$\mathbb{1}^\top y^{ss}(t) = \sum_{i=1}^{n} y_i^{ss}(t) = \Re\left\{\sum_{i=1}^{n} M_i e^{j\theta_i} \cdot e^{j\omega t}\right\} \qquad (6.11)$$

is encoded in the centroid[1]

$$Re^{j\phi} = \frac{1}{\sum_{i=1}^{n} M_i} \sum_{i=1}^{n} M_i e^{j\theta_i}, \qquad \sum_{i=1}^{n} M_i > 0, \qquad (6.12)$$

1: If all oscillators have the same fixed amplitude (such is the case for Kuramoto oscillators) the centroid reduces to $Re^{j\phi} = \frac{1}{n}\sum_{i=1}^{n} e^{j\theta_i}$ and R is known as the order parameter.

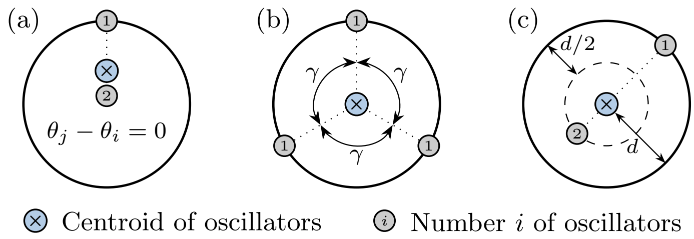

Figure 6.1: Steady-state configurations of a network of three oscillators: (a) not desynchronization, (b) desynchronization with same amplitudes, (c) desynchronization with different amplitudes.

where $R \in \mathbb{R}_+$ represents the phase-coherence of the population of oscillators and $\phi \in \mathbb{S}^1$ indicates the average phase. We note that by definition the trivial case of all oscillators with zero amplitude is ruled out; moreover, in such case the centroid in eq. (6.12) is not well defined.

The goal is to prove that the local protocol proposed in the previous section, achieves desynchronization in a network of diffusively coupled harmonic oscillators in the sense shown next.

Definition 6.3.1 (Desynchronization measure) *Consider a network of n identical oscillators* (6.8). *The network is said to achieve desynchronization if*

$$R = 0 \Leftrightarrow \mathbb{1}^\top y^{ss}(t) = 0, \tag{6.13}$$

i.e., the collective steady-state output dynamics (6.11) *is non-null with zero mean or, equivalently, the centroid* (6.12) *is at the origin of the complex plane.*

In the light of the above definition, consider a simple yet illustrative example of a network of three oscillators. Fig. 6.1 depicts the following configurations: (a) all phase differences are null, then oscillators are not desynchronized, regardless of their amplitude; (b) amplitudes are equal and the phase differences are $\theta_1 - \theta_2 = \frac{2\pi}{3}$, $\theta_2 - \theta_3 = \frac{2\pi}{3}$, then the oscillators are desynchronized, this configuration is referred in the literature as a *splay state*; (c) if $M_1 = 2M_2 = 2M_3$ and $\theta_2 = \theta_3 = \theta_1 + \pi$, then the oscillators are not in a splay state but they are desynchronized, according to the general definition we propose.

The same problem in networks of Kuramoto oscillators reduces to the so-called phase balancing problem [66] (oscillators are uniformly distributed over the unit circle) which has been solved for complete graphs with a static controller [205] and balanced graphs with a dynamic controller [201]. It is easy and relevant to notice that achieving phase balancing is neither necessary

[66] Dörfler and Bullo (2014), 'Synchronization in complex networks of phase oscillators: A survey'.

[205] Sepulchre et al. (2007), 'Stabilization of planar collective motion: All-to-all communication'.

[201] Scardovi et al. (2007), 'Synchronization and balancing on the N-torus'.

nor sufficient for desynchronization in networks of harmonic oscillators.

Local control protocol

It is known [203, 235] that a network of identical harmonic oscillators (6.8) under the diffusive coupling (6.10) achieves synchronization if the interconnection graph is connected and symmetric. This is due to the existence of the eigenvector $\mathbb{1}$ associated with the null eigenvalue of the Laplacian matrix L, which drives the oscillators toward a consensus state. On the same line of thought, in the previous section we focused our attention on the design of a state feedback matrix with a zero mean eigenvector, the Fiedler vector, associated with the null eigenvalue. The same protocol employed as a local control feedback for harmonic oscillators results in a closed-loop state matrix with a pair of imaginary conjugate eigenvalues associated with the Fiedler eigenvector thus canceling out the synchronization effect of the diffusive coupling and steering the network of oscillators to a non-null but zero mean state, thus achieving desynchronization according to Definition 6.3.1.

[203] Scardovi and Sepulchre (2009), 'Synchronization in networks of identical linear systems'.
[235] Xia and Scardovi (2016), 'Output-feedback synchronizability of linear time-invariant systems'.

Theorem 6.3.1 (Desynchronization of harmonic oscillators) *Consider a network of n identical harmonic oscillators* (6.9) *coupled with the diffusive coupling* (6.10) *and driven by the control law*

$$u_i^c(t) = -\alpha v_i(t) + \lambda_{L,2} y_i(t). \tag{6.14}$$

where $v_i(t) \in \mathbb{R}$ is a dynamic average estimation given in (6.5). *If $\mathcal{G}$ is connected and*

$$\beta = \alpha > \max\left\{\lambda_{L,2}, \frac{\omega^2}{2\lambda_{L,2}}\right\}, \qquad K_I = \sqrt{\frac{2\alpha^2 + \omega^2}{2\lambda_{L,2}}}, \tag{6.15}$$

then the network achieves desynchronization as in Definition 6.3.1 for almost all initial conditions.

Proof. Let $w_i(t) = \begin{bmatrix} y_i(t) & v_i(t) & z_i(t) \end{bmatrix}^\top \in \mathbb{R}^{3n}$ denote the augmented state of each oscillator and $w(t) = \begin{bmatrix} w_1(t) & \dots & w_n(t) \end{bmatrix}^\top$ denote the full state of the network. The network of coupled harmonic oscillators (6.8) subject to the diffusive coupling (6.10) and the feedback control (6.14) can be written in compact form as

$$\dot{w}(t) = \overbrace{\Big[(I \otimes A^*) - (L \otimes B^*)\Big]}^{M} w(t) \tag{6.16}$$

$$y(t) = (I \otimes C^*) w(t), \tag{6.17}$$

where the operator $\otimes$ denotes the Kronecker product and

$$A^* = \begin{bmatrix} A + \lambda_{L,2} BC & -\alpha B & \mathbb{0} \\ C(A + (\lambda_{L,2} + \alpha) I_2) & -2\alpha & 0 \\ \mathbb{0} & K_I & 0 \end{bmatrix},$$

$$B^* = \begin{bmatrix} BC & \mathbb{0} & \mathbb{0} \\ C & 0 & K_I \\ \mathbb{0} & 0 & 0 \end{bmatrix}, \quad C^* = \begin{bmatrix} C & 0 & 0 \end{bmatrix}.$$

Since L is symmetric, there exists an orthogonal matrix P such that $\Lambda = P^\top LP = \text{diag}([\lambda_{L,1}, \dots, \lambda_{L,n}])$ is a diagonal matrix. Consider the coordinate change

$$\widetilde{w}(t) = \widetilde{P} w(t) = \Big[P \otimes I_4\Big] w(t), \tag{6.18}$$

$$\dot{\widetilde{w}}(t) = \widetilde{M} \widetilde{w}(t) \Big[\widetilde{P}^T M \widetilde{P}\Big] \widetilde{w}(t), \tag{6.19}$$

where, by exploiting the properties of the Kronecker product,

$$\widetilde{M} = \Big[(I \otimes A^*) - (\Lambda \otimes B^*)\Big],$$

Clearly, matrices M and $\widetilde{M}$ share the same spectrum and since matrix $\widetilde{M}$ is a block diagonal matrix with blocks $\widetilde{M}_i$ given by

$$\widetilde{M}_i = A^* - \lambda_{L,i} B^* \qquad \forall i \in V,$$

then the eigenvalues of M are the eigenvalues of the blocks $\widetilde{M}_i$. We now show that, under the condition in eq. (6.15), all the eigenvalues of blocks $\widetilde{M}_i$ are strictly inside the left half of the Gauss plane, except for block $\widetilde{M}_1$, which has a zero eigenvalue, and block $\widetilde{M}_2$, which has a pair of imaginary conjugate eigenvalues.

The characteristic polynomial of each block $\widetilde{M}_i$ is

$$\begin{aligned} &\lambda^4 + a_i\lambda^3 + b_i\lambda^2 + c_i\lambda + d_i = 0 \qquad \forall i \in V \\ &a_i = (2\alpha - \lambda_{L,2}) + \lambda_{L,i} \\ &b_i = \alpha(\alpha - \lambda_{L,2}) + \lambda_{L,i}(\alpha + K_I^2) + \omega^2 \\ &c_i = \alpha\omega^2 + K_I^2\lambda_{L,i}(\lambda_{L,i} - \lambda_{L,2}) \\ &d_i = K_I^2\lambda_{L,i}\omega^2. \end{aligned} \tag{6.20}$$

It is clear that for $i = 1$ then $d_i = K_I^2\lambda_{L,1}\omega^2 = 0$ and thus the block matrix $\widetilde{M}_1$ has a zero eigenvalue. Now, by the Routh criteria, arrange the coefficients of the polynomial, and values subsequently calculated from them, as shown below

$$\begin{array}{c|ccc} \lambda^4 & 1 & b_i & d_i \\ \lambda^3 & a_i & c_i & \\ \lambda^2 & p_i & d_i & \\ \lambda^1 & q_i & & \\ \lambda^0 & d_i & & \end{array} \qquad \text{with} \quad \begin{array}{rl} p_i & = \frac{a_i b_i - c_i}{a_i} \\ q_i & = \frac{p_i c_i - a_i d_i}{p_i} \end{array}, \quad \forall i \in V. \tag{6.21}$$

A necessary and sufficient condition for all roots of (6.20) to be located in the left-half plane is that all the polynomial coefficients a_i, b_i, c_i, d_i, are positive and all of the coefficients in the first column p_i, q_i are positive. All these coefficients but q_i are positive for any $i \in V$ if

$$\alpha > \lambda_{L,2}, \quad \forall K_I > 0. \tag{6.22}$$

As far as concerns the coefficients q_i, if

$$\alpha > \frac{\omega^2}{2\lambda_{L,2}}, \qquad K_I = \sqrt{\frac{2\alpha^2 + \omega^2}{2\lambda_{L,2}}}. \tag{6.23}$$

then $q_i = 0$ for $i = 2$ and $q_i > 0$ otherwise. Thus, the block matrix $\widetilde{M}_2$ has a pair of imaginary conjugate eigenvalues. Moreover, conditions in (6.22) and (6.23) are equivalent to condition (6.15).

Exploiting the change of variable (6.18), one can write the state evolution of the system as

$$w(t) = e^{Mt}w(0) = e^{\widetilde{P}\widetilde{M}\widetilde{P}^\intercal t}w(0) = \widetilde{P}e^{\widetilde{M}t}\widetilde{P}^\intercal w(0).$$

and so the output can be written as

$$y(t) = (I_n \otimes C^*)w(t) = (I_n \otimes C^*)\widetilde{P}e^{\widetilde{M}t}\widetilde{P}^\intercal w(0).$$

As $t \to \infty$, all blocks $e^{\widetilde{M}_i t}$ for $i = 3, \ldots, n$ tend to zero because of the negative real part of their eigenvalues. Thus, since the columns of P are the eigenvectors of matrix L it follows

$$\lim_{t\to\infty} y(t) = (I_n \otimes C^*)\left[v_{L,1}v_{L,1}^\top \otimes e^{\widetilde{M}_1 t} + v_{L,2}v_{L,2}^\top \otimes e^{\widetilde{M}_2 t}\right]w(0)$$
$$= \left[(v_{L,1}v_{L,1}^\top \otimes C^* e^{\widetilde{M}_1 t}) + (v_{L,2}v_{L,2}^\top \otimes C^* e^{\widetilde{M}_2 t})\right]w(0).$$

As previously shown, matrix $\widetilde{M}_1$ has only one null eigenvalue (the others have negative real part) with eigenvector $\mathbb{1}_5 \otimes [0\,0\,0\,1]^\top$, thus

$$\lim_{t\to\infty} e^{\widetilde{M}_1 t} \propto \begin{bmatrix} \mathbb{0}_{3\times3} & \mathbb{0}_{3\times1} \\ \mathbb{0}_{1\times4} & 1 \end{bmatrix}.$$

Since

$$\lim_{t\to\infty} C^* e^{\widetilde{M}_1 t} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbb{0}_{3\times3} & \mathbb{0}_{3\times1} \\ \mathbb{0}_{1\times4} & 1 \end{bmatrix} = \mathbb{0}$$

it follows

$$\lim_{t\to\infty} y(t) = \left[v_{L,2}v_{L,2}^\top \otimes C^* e^{\widetilde{M}_2 t}\right]w(0).$$

The collective steady-state output dynamics results in

$$\mathbb{1}^\top y^{ss}(t) = \lim_{t\to\infty} \mathbb{1}^\top y(t)$$
$$= \mathbb{1}^\top \left[v_{L,2}v_{L,2}^\top \otimes C^* e^{\widetilde{M}_2 t}\right]w(0)$$
$$= \left[\mathbb{0} \otimes C^* e^{\widetilde{M}_2 t}\right]w(0) = \mathbb{0}$$

We proved that the network reaches a steady-state with a zero mean state output dynamics, in which each oscillator has a non-trivial oscillatory behavior due to the pair of imaginary conjugate eigenvalues of block $\widetilde{M}_2$, and so of M. This completes the proof. □

6.4 Simulations and discussion

Fiedler vector estimation

In the first simulation, a MAS with $n = 7$ agents (6.2), line topology and local control law (6.4)-(6.5) is considered. Accordingly with conditions of Theorem 6.2.2, we chose the gain for the eigenvector estimator as α and the gains for the integral dynamic average consensus estimator as β, K_I as follows

$$\alpha = 10, \quad \beta = 20, \quad K_I = 100.$$

All the agents' state variables are randomly initialized in the interval. In Fig. 6.2 it is shown how the MAS converges to a scaled Fiedler vector, i.e.,

$$\lim_{t\to\infty} y(t) = \gamma v_{L,2} = \widetilde{v}_{L,2},$$

where γ denotes the scale factor

$$\gamma = \lim_{t\to\infty} ||y(t)|| = 0.807.$$

In the second simulation, we compare our protocol with the one in [241] because of their similar structure; a MAS with $n = 5$ agents (6.2) and local control law (6.4) is considered. We keep our notation for any common variable (i.e., α, β and K_I) while we use the notation in [241] for the remaining variables (i.e., k_2, k_3 and K_P). Accordingly with conditions of Theorem 6.2.2, we chose the gain for the eigenvector estimator as α and the gains for the integral dynamic average consensus estimator as β, K_I as follows

$$\alpha = 6, \quad \beta = 25, \quad K_I = 10.$$

The additional gains for the algorithm in [241] are chosen as $k_2 = 1$, $k_3 = 20$ and $K_P = 50$. We point out that the choices of network topology and free parameters are the same of Example 1 in [241]. Moreover, the state variables common to the algorithms are initialized to the same values while the extra state values of the algorithm in [241] are chosen in order to nullify the initial error estimation, thus guaranteeing a fair comparison. In Fig. 6.3 it is shown the estimation error

$$e(t) = ||y(t) - \widetilde{v}_{L,2}||,$$

for both protocols, revealing that the proposed protocol provides a faster exponential convergence rate.

In the third and last simulation, we consider a MAS with increasing number n of agents, line topology and local control law (6.4). With the choice of the line topology we are considering the worst case scenario for the dynamic average estimator. According to conditions of Theorem 6.2.2, we chose the gain for the eigenvector estimator as α and the gains for the integral dynamic average consensus estimator as β, K_I as follows

$$\alpha = 2\lambda_{L,2}, \quad \beta = 10, \quad K_I = 15,$$

while the extra gain parameters of the protocol in [241] are chosen

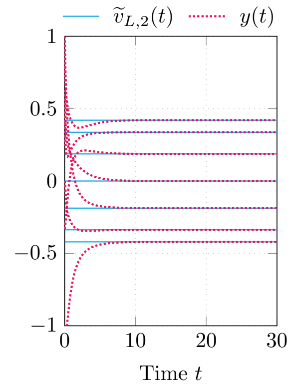

Figure 6.2: Fiedler vector estimation error in a line network of 7 agents.

[241] Yang et al. (2010), 'Decentralized estimation and control of graph connectivity for mobile sensor networks'.

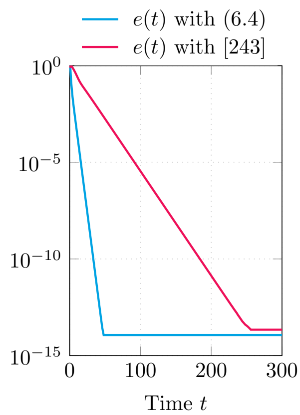

Figure 6.3: Fiedler vector estimation error in a network of 5 agents.

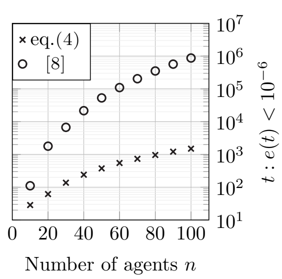

Figure 6.4: Convergence time of Fiedler vector estimation in line networks.

as $k_2 = 1$, $k_3 = 2\lambda_{L,2}$ and $K_P = 25$. Fig. 6.4 shows, for different values of n, the time required by the two algorithms to reach an error of the order of 10^{-6}. The simulation confirms that, in the face of the assumption on the knowledge of the algebraic connectivity, the proposed protocol has a faster convergence rate, making it more scalable with the number of the agents in the network.

Desynchronization of harmonic oscillators

There are several research works focused on dynamics and control of some typical coupled oscillator systems derived from some practical engineering problems. In particular, synchronization of networked mechanical oscillator systems have been subject of interest [151, 164, 192, 216]. Here we give a physical example of application of Theorem 6.3.1.

Consider the networked mechanical systems consisting of n train wagons of identical mass m with linear dumper b interaction [151] and a mass-spring-damper modeling the interaction with the ground. Assuming a (ideally) null damping in order to guarantee the best comfort to the passengers, and denoting the spring coefficient with k, the model becomes the one in (6.8)-(6.10) with natural frequency $\omega = \sqrt{k/m} = 0.1$ and dumping coefficient $b = 1$. The feedback control (6.14)-(6.5) models the active suspensions between wagons and desynchronization corresponds to the minimum stress on the rails since the sum of the forces becomes as the time passes.

Simulation of a line-topology network with $n = 5$ nodes is shown in Fig. 6.5 with $\alpha = 1.88$ and $K_I = 2$, according to Theorem 6.3.1. The oscillators start evolving without any coupling until at $t = 5$

[192] Ren (2008), 'Synchronization of coupled harmonic oscillators with local interaction'.

[216] Su et al. (2009), 'Synchronization of coupled harmonic oscillators in a dynamic proximity network'.

[164] Nair and Leonard (2008), 'Stable synchronization of mechanical system networks'.

[151] Mei and Goodall (2002), 'Use of multiobjective genetic algorithms to optimize inter-vehicle active suspensions'.

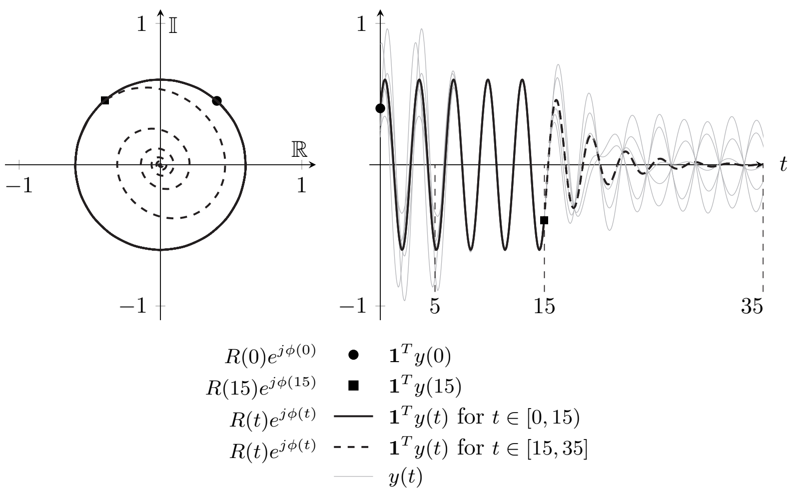

Figure 6.5: Evolution of 5 coupled harmonic oscillators. The diffusive coupling (6.10) is activated at $t = 5$, while the desynchronizing local feedback (6.14)-(6.5) is activated at $t = 15$.

the diffusive coupling is enabled and synchronization is achieved. Let us denote with $R(t)e^{j\phi(t)}$ the centroid as defined in (6.11)-(6.12) given $x(t)$ as the initial condition of the network $x(t)$; it is clear that $\lim_{t\to\infty} R(t) = R$. Thus, as can be seen in Fig. 6.5, the collective output dynamics has constant module $R(t) = 0.6$ for $t \in [0, 15)$. At $t = 15$ the proposed control feedback is activated and the network is shown to reach desynchronization in the sense of Definition 6.3.1, i.e., $R = \lim_{t\to\infty} R(t) = \lim_{t\to\infty} \frac{1}{n}\mathbb{1}^\top y(t) = \mathbb{1}^\top y^{ss}(t) = 0$.

Dynamic Consensus on the min/max value

7

In this chapter we are mainly concerned with the problem of estimating the maximum (or, alternatively, the minimum) value among a set of external reference signals given as input u_i to the agents in discrete-time

$$x_i(k+1) = f(u_i(k), x_i(k), x_j(k) : j \in \mathcal{N}_i),$$

in a distributed way by designing the local interaction rule $f(\cdot)$. The main working assumption and the formalization of the dynamic min/max-consensus problem are provided in Section 7.2, while the related literature is reviewed in Section 7.1.

After having shown how the simple max-consensus protocol [217] is biased in non-autonomous systems and can not be applied to solve tracking problems, in Section 7.3 we propose two strategies to modify the popular protocol in [217] for solving the static min/max-consensus by either 1) introducing a design parameter or 2) by augmenting the state space of each agent, making also the new protocols robust to re-initialization; these protocols are presented in Section 7.3. The first protocol achieves bounded steady-state and tracking error which can be traded-off for improved convergence time by tuning the newly added protocol parameter. The second protocol achieves zero steady-state error and bounded tracking error and requires knowledge of an upper bound to the diameter of the graph representing the network to be executed. We apply the proposed dynamic max-consensus problem in the scenario of open MASs where agents can join and leave the network during the algorithm execution and solve for the first time a distributed size estimation problem for an anonymous network with time-varying size in Section 7.4. Finally, in Section 7.5 numerical simulations to corroborate the theoretical analysis are provided.

[217] Tahbaz-Salehi and Jadbabaie (2006), 'A one-parameter family of distributed consensus algorithms with boundary: From shortest paths to mean hitting times'.

7.1 Related literature

Dynamic average consensus

Most of the literature usually considers consensus problems for autonomous MASs, i.e., agents converge to a state value which is function of the initial state of the network. On the contrary, in the dynamic consensus problem the agents are assumed to be non-autonomous and are requested to converge to a state value which is a function of the time-varying reference signals given as input to the agents, such as the average [120], the median value [227], the min/max value and so on. The dynamic consensus problem is related to online optimization, namely the problem of tracking the optimal solution of a time-varying optimization problem [**Simonetto17**]. The main difference between these two problems is that the optimal value of the objective function is usually unknown for online optimization thus usually forcing to employ the gradient of the objective function as a local interaction rule, while it is known in the dynamic consensus problem and more specific and performing protocols can be designed to solve the problem.

[120] Kia et al. (2019), 'Tutorial on Dynamic Average Consensus: The Problem, Its Applications, and the Algorithms'.

[227] Vasiljevic et al. (2020), 'Dynamic Median Consensus for Marine Multi-Robot Systems Using Acoustic Communication'.

[**Simonetto17**] **Simonetto17** (**Simonetto17**), **Simonetto17**.

The literature has focused significantly on the dynamic average-consensus problem and an insightful tutorial has been provided by Kia et al. [120]. The pioneering work addressed the continuous-time case by considering the derivative of the reference signals to design the dynamics of the estimator [213], while the discrete-time case has been addressed some years after by considering the differences of the reference signals instead [254]. Because they rely on the computation of inputs derivatives or differences, both algorithms suffer from a persistent bias due to re-initialization errors in the case of changes in the network during the protocol execution, as well as their vulnerability to noise. Since then, many authors proposed variants of dynamic consensus protocols. In [121], the problem was solved by an event-triggered approach with intermittent communications. In [254], a discrete-time dynamic average consensus was proposed by exploiting a model of the inputs by considering their finite differences of n-th order. In [159], the authors extended the approach in [254] by proposing a variant of the discrete-time dynamic average consensus protocol, guaranteeing robustness against re-initialization errors. In [82], the authors investigated the performance of a cascade of proportional dynamic consensus in discrete-time and randomized versions in order to address robustness with respect to re-initialization error and tunable steady-state and maximum tracking error. Many other dynamic average consensus protocols can be found in [20, 78, 118, 204, 224], with approaches that address robustness to re-initialization and steady-state error under various assumptions on the network topology and the scenario where agents may join and leave the network. Other recent works propose robust dynamic consensus protocols in continuous-time [92] and consider graph balancing

[120] Kia et al. (2019), 'Tutorial on Dynamic Average Consensus: The Problem, Its Applications, and the Algorithms'.

[213] Spanos et al. (2005), 'Dynamic consensus on mobile networks'.

[254] Zhu and Martínez (2010), 'Discrete-time dynamic average consensus'.

strategies for dynamic consensus in directed graphs [138]. For further references, the interested reader is referred to the recent tutorial in [120].

The min/max consensus problem

While the estimation of average values has received a lot of attention in the MAS literature, it is far from being the the only problem of practical relevance and theoretical significance. In particular, the development of a dynamic consensus protocol which achieve consensus and track the value of the maximum or, alternatively, the minimum value among the exogenous reference signals, is missing in the actual literature. The importance of the problem is evident from the large number of existing applications of the static max/min-consensus problem, which include distributed synchronization, such as time-synchronization [57] and target tracking [186], and network parameter estimation, such as the cardinality [143] and highest/lowest node degree [31]. The so-called max-consensus problem has been thoroughly investigated. Its objective is to make the states of a network of agents converge to the maximum of their initial states [175]. First protocols solving the max-consensus problem have been proposed by Cortes [45] and by Tahbaz and Jadbabaie [217], in continuous-time and discrete-time frameworks, respectively. The popular discrete-time protocol in [217] requires that the agents update their state at each iteration by taking the maximum among the state values of the neighbors and their own state. This protocol was also characterized for the max-consensus problem in [166], where conditions to achieve max-consensus and convergence rate are derived in a max-plus algebraic setting. Other approaches include soft-max estimators [217, 249] and gossip based or randomized approaches [1, 109]. The max-consensus problem has been addressed in more general scenarios. Convergence results for time-varying networks with synchronous switching topologies have been investigated in [167]. Asynchronous updates and communications affected by time-delays have been considered in [94], while a stochastic framework for asynchronous updates has been proposed in [109]. The effect of noise in communications among the agents has been characterized in [161, 250]. Finally, in the context of open MASs where the size of the network is time-varying, a gossip algorithm for max-consensus has been investigated in [1].

[175] Olfati-Saber and Murray (2004), 'Consensus problems in networks of agents with switching topology and time-delays'.

[45] Cortés (2008), 'Distributed algorithms for reaching consensus on general functions'.

[217] Tahbaz-Salehi and Jadbabaie (2006), 'A one-parameter family of distributed consensus algorithms with boundary: From shortest paths to mean hitting times'.

Open network and size estimation

Since our main application considers networks wherein agent are allowed to join or leave, the so-called open networks, we provide here a brief review.

In this recent topic of research, interesting contributions can be found in [79, 80, 100, 101, 226] where the authors formulate consensus and dynamics consensus problems for networks of time-varying size. The works in [100, 101, 226] consider stochastic arrivals and departures of agents in the network while [79, 80] does not consider a model for agent arrivals and departures from the network. An interesting problem in open networks is the estimation of the actual number of active agents, which is precisely a time-varying quantity to be estimated. The problem is quite straightforward to solve in the case it is possible to assign unique IDs to the agents and their are able to share it with their neighbors, since a trivial broadcast and count technique would be effective [132, 206]. The case in which the identity of the node must be preserved is much more challenging and falls in the framework of the so-called anonymous networks side[238].

The size estimation problem in anonymous networks counts a high number of interesting applications, e.g., maintenance purposes in ad-hoc wireless sensor networks [39], optimization of query access plans in internet-scale data networks [171], coordination of robotic agents [33], and so on.

A fruitful strategy is based on generating random variables at each network's node and on the subsequent functional estimation [24]. The result of such an estimation step is used to infer the number of nodes in the network in a distributed way. In the approach proposed in [24] the network topology is assumed to be unknown to the agents; however, as often happens, some mild knowledge on the network structure is assumed to be known, in particular an upper bound on the diameter of the network is known is usually required to be known. In [207], max consensus is first used to distributedly decide a leader in the graph and then exploit this fact for node counting. Along this line of thought, consensus based strategy has been proposed, as the one in [225], which makes use of statistical inference methods to estimate the number of nodes who took part in the random generation of numbers by estimating the average or maximum number. Similarly, in [250] the problem is solved by means of norm estimation and average consensus in the presence of communication noise. In [60] we exploit our novel dynamic max-consensus algorithm to enable the tracking of the

[24] Bawa et al. (2003), *Estimating aggregates on a peer-to-peer network.*

[207] Shames et al. (2012), 'Distributed network size estimation and average degree estimation and control in networks isomorphic to directed graphs'.

[225] Varagnolo et al. (2014), 'Distributed Cardinality Estimation in Anonymous Networks'.

[60] Deplano et al. (2020), 'Dynamic Min and Max Consensus and SizeEstimation of Anonymous Multi-Agent Networks'.

time-varying network's size thus avoiding the need for network wide re-initialization of the distributed estimation procedure.

7.2 Dynamic min/max-consensus problem

We consider a network of n agents modeled as discrete-time dynamical systems with state $x_i(k) \in \mathbb{R}^m$, each of which has access to time-varying external reference signal $u_i(k) \in \mathbb{R}$ and interacts with other agents according to a graph $\mathcal{G}$ and a local interaction protocol

$$x_i(k) = f_i(u_i(k), x_j(k-1) : j \in \mathcal{N}_i^\circ). \tag{7.1}$$

The output of each agent is defined as

$$y_i(k) = g_i(x_i(k)).$$

The *dynamic min/max-consensus problem* consists in the design of a local interaction protocol $f_i(\cdot)$ and an output function $g_i(\cdot)$ which steers the agents' output $y_i(k) \in \mathbb{R}$ to track the maximum $\overline{u}(k) \in \mathbb{R}$ or the minimum $\underline{u}(k) \in \mathbb{R}$ among the time-varying reference signals.

Next, we state a common assumption in the dynamic consensus literature used also in this paper, i.e., the boundedness of the change of the reference signals in a time-window $[k, k+T]$, with $T \in \mathbb{N}$, which is defined as

$$\Delta u_i(k, T) = u_i(k) - u_i(k-T), \quad \forall i \in V. \tag{7.2}$$

Assumption 7.2.1 *The maximum absolute change*[1] *of the reference signals in one $T = 1$ instant is bounded by a constant $\Pi \in \mathbb{R}_+$, i.e.,*

$$|\Delta u_i(k, 1)| \leq \Pi, \quad \forall i \in V, \quad \forall k \geq 0. \tag{7.3}$$

1: Note that if the reference signals are sampled versions of continuous-time signals, then by increasing the sampling frequency their change in one iteration is reduced. Thus, for any signal with bounded change there exists a sampling frequency such that Assumption 7.2.1 is also satisfied.

In a similar way, we define the change of the maximum and the minimum, respectively, among the reference signals as

$$\begin{aligned} \overline{\Delta u}(k, T) &= \overline{u}(k) - \overline{u}(k-T), \\ \underline{\Delta u}(k, T) &= \underline{u}(k) - \underline{u}(k-T). \end{aligned} \tag{7.4}$$

In this paper we also deal with the scenario of an open multi-agent system (OMAS), wherein the agents may leave or join, thus resulting in a network's change in terms of number of active agents and active communication channels. These changes in the network are encoded into a time-varying graph $\mathcal{G}(k) = (V(k), E(k))$ describing the interconnection among the $n(k)$ active agents. As soon as such a change occurs, the new agents' reference signals can be possibly much larger or smaller than those of the agents previously connected to the network. Thus, to address open networks we assume that the frequency at which the agents can join or leave the network is bounded as formalized next.

Assumption 7.2.2 *There exists a minimum dwell time $\Upsilon \in \mathbb{N}$ between two consecutive changes of the graph $\mathcal{G}$.*

The main objective of this work is to provide two local interaction protocols (7.1) that solve the dynamic min/max-consensus problem described above in MAS. The proposed protocols are shown to be robust to re-initialization, or, in other words, their execution does not need to be interrupted and the state of the agents re-initialized after the network changes in topology or size, thus enabling their use in open MAS. We characterize the performance of the proposed protocols in terms of convergence time as well as tracking error

$$\overline{e}(k) = \max_{i \in V} |y_i(k) - \overline{u}(k)|, \quad \underline{e}(k) = \max_{i \in V} |y_i(k) - \underline{u}(k)|. \tag{7.5}$$

7.3 Proposed distributed and unbiased protocols

The popular protocol [94, 161, 166, 217] which solves the min/max-consensus problem makes use of the following local interaction rules, respectively,[2]

$$x_i(k) = \max_{j \in \mathcal{N}_i^{\circ}} \{x_j(k-1)\}, \tag{7.6}$$

$$x_i(k) = \min_{j \in \mathcal{N}_i^{\circ}} \{x_j(k-1)\}, \tag{7.7}$$

with $x_i \in \mathbb{R}$. These protocols enable the agents' states to converge to the maximum/minimum among the initial state. Thus, assuming a set of constant reference signals $u_i(k) = u_i(0) \in \mathbb{R}$ for $k \in \mathbb{N}$, the protocols estimate their maximum by requiring the following initialization step

$$x_i(0) = u_i(0). \tag{7.8}$$

[217] Tahbaz-Salehi and Jadbabaie (2006), 'A one-parameter family of distributed consensus algorithms with boundary: From shortest paths to mean hitting times'.

[166] Nejad et al. (2009), 'Max-consensus in a max-plus algebraic setting: The case of fixed communication topologies'.

[94] Giannini et al. (2016), 'Asynchronous max-consensus protocol with time delays: convergence results and applications'.

[161] Muniraju et al. (2019), 'Analysis and design of robust max consensus for wireless sensor networks'.

2: Note that in this chapter it is used $\mathcal{N}_i^{\circ}$ to denote the set of neighbors of agent i with i itself, i.e., $\mathcal{N}_i^{\circ}$.

A naive generalization of these protocols to deal with time-varying inputs could be

$$x_i(k) = \max_{j \in \mathcal{N}_i^\circ} \{x_j(k-1), u_i(k)\}. \tag{7.9}$$

$$x_i(k) = \min_{j \in \mathcal{N}_i^\circ} \{x_j(k-1), u_i(k)\}. \tag{7.10}$$

For static inputs the protocols in eq. (7.9)-(7.10) are equivalent to those in eq. (7.6)-(7.7). These protocols are biased in the following sense:

- The max-consensus protocol in eq. (7.9) provides a consensus value which is non-decreasing and thus it cannot track the max value of arbitrary time-varying signals that decrease in value;
- The min-consensus protocol in eq. (7.10) provides a consensus value which is non-increasing and thus it cannot track the min value of arbitrary time-varying signals that increase in value.

Figure 7.1 shows the evolution of a random network of agents executing protocols in eq. (7.9)-(7.10) with time-varying reference signals and without any re-initialization logic. As one cannotice, the tracking is lost every time the maximum or minimum among the reference signals is below or above the agents' states, respectively. Therefore, only a re-initialization of the protocol execution in the whole network can mitigate such an estimation bias for time-varying reference signals, which is a significant drawback for the implementation of a distributed algorithm in large-scale networks.

We propose two different approaches to modify the conventional protocols in eq. (7.9)-(7.10) in order to overcome the issue of

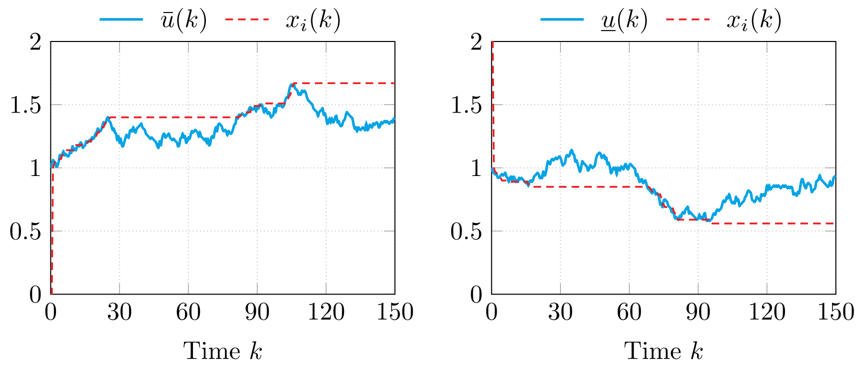

Figure 7.1: Biased behavior of conventional max-consensus protocol in eq. (7.9) and min-consensus protocol in eq. (7.10).

re-initializing the network and thus allowing the tracking of time-varying reference signals:

1. The first approach enables the agents to converge to an *approximate* consensus on the min/max value without requiring any information about the network graph. The two protocols are named:
   - *Dynamic Max-Consensus* (DMAC) Protocol;
   - *Dynamic min-Consensus* (DMIC) Protocol.
2. The second approach enables the agents to reach an *exact* consensus on the min/max value by requiring the knowledge of an upper bound of the network's graph diameter. The two protocols are named:
   - *Exact Dynamic Max-Consensus* (EDMAC) Protocol;
   - *Exact Dynamic min-Consensus* (EDMIC) Protocol.

Remark 7.3.1 All protocols proposed in this section are robust to re-initialization. This means that if a change in the network happens, such as a variation of the topology or the number of active nodes, there is no need to re-initialize the state of the agents to guarantee the tracking of the min/max value. Clearly, the change may results in an unpredictable discontinuity of the quantity of interests, thus a certain amount of time, called the convergence time, to re-establish an effective tracking is required. Their robustness enables their employment in time-varying and/or open networks, as in the case of our main application.

Dynamic min/max-consensus (DMAC/DMIC) protocols

In this section we consider networks of agents with scalar state $x_i \in \mathbb{R}$. The strategy proposed in this section suggests integrating an additive parameter α in the generalized protocol in eq. (7.9)-(7.10) allowing non monotonic estimation behavior. The proposed local interaction rules become the following

$$x_i(k) = \max_{j \in \mathcal{N}_i^\circ} \left\{ x_j(k-1) - \alpha, u_i(k) \right\} \tag{7.11}$$

$$x_i(k) = \min_{j \in \mathcal{N}_i^\circ} \left\{ x_j(k-1) + \alpha, u_i(k) \right\} \tag{7.12}$$

where $\alpha \in \mathbb{R}_+$ is a scalar tuning parameter and the output is,

$$y_i(k) = x_i(k),$$

which represents the current estimated value. Before stating the main theorem characterizing these protocols, we need two useful lemmas; these lemmas provides, respectively, an upper bound to the maximum among the states and a lower bound to the minimum of the states.

Lemma 7.3.1 *Consider a MAS wherein agents evolve according to* (7.11)*, i.e.,*

$$x_i(k) = \max_{j \in \mathcal{N}_i^\circ} \left\{ x_j(k-1) - \alpha, u_i(k) \right\},$$

and assume Assumption 7.2.1 to be in force. For any arbitrary initialization $x_i(0) \in \mathbb{R}$ with $i \in V$, and any generic initial instant of time $k_0 \in \mathbb{N}$, if graph $\mathcal{G}$ is connected and if $\alpha > \Pi$ then the maximum among the inputs is upper bounded by

$$\overline{x}(k) = \overline{u}(k), \quad k \geq k_0 + T', \tag{7.13}$$

with $\overline{x}(k) = \max_{i \in V} x_i(k)$ and

$$T' = \frac{\max\left\{ \underline{y}(k_0) - \overline{u}(k_0), 0 \right\}}{\alpha - \Pi}. \tag{7.14}$$

Proof. At time $k \geq k_0 + T \geq 1$, it holds

$$\begin{aligned}
\overline{x}(k) &= \max_{i \in V} x_i(k) \\
&= \max_{i \in V} \max_{j \in \mathcal{N}_i^\circ} \{x_j(k-1) - \alpha, u_i(k)\} \\
&= \max_{i \in V} \{x_i(k-1) - \alpha, u_i(k)\} \\
&= \max\{\overline{x}(k-1) - \alpha, \overline{u}(k)\} \\
&= \max\{\overline{x}(k-1) - \alpha, \overline{u}(k-1) + \overline{\Delta u}(k,1)\} \\
&\geq \max\{\overline{x}(k-2) - 2\alpha, \overline{u}(k-2) + \overline{\Delta u}(k,2)\} \\
&\geq \qquad \vdots \\
&\geq \max\{\overline{x}(k_0) - T\alpha, \overline{u}(k_0) + \overline{\Delta u}(k,T)\} \\
&\geq \max\{\overline{x}(k_0) - T\alpha, \overline{u}(k_0) - T\Pi\}
\end{aligned}$$

where the last inequality follows from Assumption 7.2.1, in fact

$\overline{\Delta u}(k, T) \in [-T\Pi, T\Pi]$. Due to condition (7.16), the maximum change of the inputs Π is smaller than α, therefore there exists a time after which the system "reaches" the input, i.e.,

$$\overline{x}(k_0) - T\alpha < \overline{u}(k_0) - T\Pi.$$

Solving for T, we obtain T' as in (7.14). Thus, for $k \geq T'$, it holds $\overline{x}(k) = \overline{u}(k)$,, proving the veracity of eq. (7.13). □

Lemma 7.3.2 *Consider a MAS wherein agents evolve according to* (7.11), *i.e.,*

$$x_i(k) = \max_{j \in \mathcal{N}_i^\circ} \left\{ x_j(k-1) - \alpha, u_i(k) \right\},$$

and assume Assumption 7.2.1 to be in force. For any arbitrary initialization $x_i(0) \in \mathbb{R}$ with $i \in V$, and any generic initial instant of time $k_0 \in \mathbb{N}$, if graph $\mathcal{G}$ is connected and if $\alpha > \Pi$ then the minimum among the inputs is lower bounded by

$$\underline{x}(k) \geq \overline{u}(k) + \overline{\Delta u}(k, \delta_{\mathcal{G}}) - \alpha\delta_{\mathcal{G}}, \quad k \geq k_0 - \delta_{\mathcal{G}}, \tag{7.15}$$

with $\underline{x}(k) = \min_{i \in V} x_i(k)$.

Proof. At time k_0 we define the set

$$\mathcal{V}_0 = \{i \in V : x_i(k_0) = \overline{x}(k_0)\}$$

denoting the set of agents whose state at time k_0 is the maximum among all others. Next, we define the set $\mathcal{V}_1$ as the set of one-hop neighbors of nodes in set $\mathcal{V}_0$. Formally,

$$\mathcal{V}_1 = \left\{ i \in V : (i, j) \in E, \ j \in \mathcal{V}_0 \right\}.$$

The state update rule (7.11) at $k = k_0 + 1$ for the agents belonging to this set reduces to

$$x_i(k_0 + 1) = \max\{\overline{x}(k_0) - \alpha, u_i(k_0 + 1)\}, \quad \forall i \in \mathcal{V}_1$$

because they have a neighbor $j \in V_0$ with state value $x_j(k_0) = \overline{x}(k_0)$. By induction, define the set

$$\mathcal{V}_\ell = \left\{ i \in V : (i, j) \in E, j \in \bigcup_{s=0}^{\ell-1} \mathcal{V}_s \right\},$$

Protocol 1: Dynamic Max-Consensus (DMAC)

Initialization: $x_i(0) \in \mathbb{R}$ for $i \in V$.
Init. for opt. conv. time: $x_i(0) = u_i(0)$ for $i \in V$.
Input: Tuning parameter $\alpha \in \mathbb{R}_+$.
Output: $y_i(k) = x_i(k) \in \mathbb{R}$ for $i \in V$.

for $k = 0, 1, 2, \ldots$ **each node** i **do**
  Gather $x_j(k)$ from each neighbor $j \in \mathcal{N}_i$
  Update the current state according to
  $x_i(k) = \max_{j \in \mathcal{N}_i^\circ} \left\{ x_j(k-1) - \alpha, u_i(k) \right\}$

Protocol 2: Dynamic min-Consensus (DMIC)

Initialization: $x_i(0) \in \mathbb{R}$ for $i \in V$.
Init. for opt. conv. time: $x_i(0) = u_i(0)$ for $i \in V$.
Input: Tuning parameter $\alpha \in \mathbb{R}_+$.
Output: $y_i(k) = x_i(k) \in \mathbb{R}$ for $i \in V$.

for $k = 0, 1, 2, \ldots$ **each node** i **do**
  Gather $x_j(k)$ from each neighbor $j \in \mathcal{N}_i$
  Update the current state according to
  $x_i(k) = \min_{j \in \mathcal{N}_i^\circ} \left\{ x_j(k-1) + \alpha, u_i(k) \right\}$

and for all agents in these sets the state update rule (7.11) reduces to

$$x_i(k_0 + \ell) = \max\{\overline{x}(k_0) - \ell\alpha, u_i(k_0 + \ell)\}, \quad \forall i \in \mathcal{V}_\ell.$$

Since the longest shortest path between two nodes in a connected graph is at most equal to its diameter $\delta_\mathcal{G}$, for $\ell \geq \delta_\mathcal{G}$ it holds $\mathcal{V}_\ell \equiv V$. Therefore, for $k \geq k_0 + \delta_\mathcal{G}$ it holds

$$x_i(k) = \max\{\overline{x}(k - \delta_\mathcal{G}) - \ell\alpha, u_i(k)\}, \quad \forall i \in \mathcal{V}.$$

Finally, by the trivial property of the max function and eq. (7.13) the next chain of inequalities holds,

$$\begin{aligned} \underline{x}(k) &\geq \overline{x}(k - \delta_\mathcal{G}) - \delta_\mathcal{G}\alpha \\ &\geq \overline{u}(k - \delta_\mathcal{G}) - \delta_\mathcal{G}\alpha \\ &\geq \overline{u}(k) - \overline{\Delta u}(k, \delta_\mathcal{G}) - \alpha\delta_\mathcal{G}. \end{aligned}$$

□

In Protocols 1-2 we detail the Dynamic Max-Consensus (DMAC) and the Dynamic min-Consensus (DMIC) Protocols, respectively, while in the next theorem we provide a characterization of the convergence time and the tracking error.

Theorem 7.3.3 *Consider a MAS executing either Protocol 1 or Protocol 2 under Assumption 7.2.1, with arbitrary initialization $x_i(0) \in \mathbb{R}$ for $i \in V$ and consider a generic initial instant of time $k_0 \in \mathbb{N}$. If graph $\mathcal{G}$ is connected and if*

$$\alpha > \Pi, \tag{7.16}$$

then there exists a convergence time T_c such that the tracking errors defined in eq. (7.5) are bounded for $k \geq k_0 + T_c > 0$ by

$$e(k) \leq |\alpha\delta_{\mathcal{G}} + \Delta u(k, \delta_{\mathcal{G}})| \leq (\alpha + \Pi)\delta_{\mathcal{G}}, \tag{7.17}$$

where[3] *$e(k)$ and $\Delta u(k)$ are given in (7.5) and (7.4), and it holds*

$$T_c \leq \max\left\{\delta_{\mathcal{G}}, \frac{d(k_0)}{\alpha - \Pi}\right\}, \tag{7.18}$$

where the distance $d(k_0)$ is equal to $\underline{y}(k_0) - \overline{u}(k_0)$ for Protocol 1 and to $\overline{y}(k_0) - \underline{u}(k_0)$ for Protocol 2.

3: The error $e(k)$ and the change $\Delta u(k, \cdot)$ are to be understood as the error on the max value $\overline{e}(k)$ and the change of the max value $\overline{\Delta u}(k, \dot{)}$ for Protocol 1 and the error on the min value $\underline{e}(k)$ and the change of the min value $\underline{\Delta u}(k, \dot{)}$ for Protocol 2.

Proof. We first consider the case of Protocol 1 and then the case of Protocol 2 is derived as a special case by showing its equivalence with Protocol 1.

Thus, let us first consider Protocol 1. By Lemmas 7.3.1-7.3.2, at a generic time $k_0 \in \mathbb{N}$ the maximum and the minimum among the agents' states are bounded by the following

$$\begin{aligned} \overline{x}(k) &= \overline{u}(k), & k &\geq k_0 + T', \\ \underline{x}(k) &\geq \overline{u}(k) + \overline{\Delta u}(k, \delta_{\mathcal{G}}) - \alpha\delta_{\mathcal{G}}, & k &\geq k_0 + \delta_{\mathcal{G}}, \end{aligned}$$

where T' is given by

$$T' = \frac{\max\left\{\underline{x}(k_0) - \overline{u}(k_0), 0\right\}}{\alpha - \Pi}.$$

The above relations are both satisfied for $k \geq k_0 + \max\{\delta_G, T'\}$. This proves that the convergence time is upper bounded as in eq. (7.18). Considering the output $y_i(k) = x_i(k)$, the bound on the tracking error given in (7.17) can be derived as follows

$$\begin{aligned}
\overline{e}(k) &= \max_{i\in V} |y_i(k) - \overline{u}(k)| \\
&= \max_{i\in V} |x_i(k) - \overline{u}(k)| \\
&= \max\{|\overline{x}(k) - \overline{u}(k)|, |\underline{x}(k) - \overline{u}(k)|\} \\
&= |\underline{x}(k) - \overline{u}(k)| \\
&\leq \left|\alpha\delta_{\mathcal{G}} + \overline{\Delta u}(k, \delta_{\mathcal{G}})\right| \leq (\alpha + \Pi)\delta_{\mathcal{G}},
\end{aligned}$$

where the last inequality is due to Assumption 7.2.1.

Let us now consider Protocol 2. Let $v(k) = -u(k)$ for which it holds $\overline{v}(k) = -\underline{u}(k)$ and rewrite the local interaction rule as

$$\begin{aligned}
x_i(k) &= \min_{j\in\mathcal{N}_i^\circ} \left\{x_j(k-1) + \alpha, u_i(k)\right\} \\
&= \min_{j\in\mathcal{N}_i^\circ} \left\{x_j(k-1) + \alpha, -v_i(k)\right\} \\
&= -\max_{j\in\mathcal{N}_i^\circ} \left\{-x_j(k-1) - \alpha, v_i(k)\right\}.
\end{aligned}$$

By letting $z_i(k) = -x_i(k)$, one concludes that the MAS with state $x(k)$ and input $u(k)$ executing Protocol 2 and estimating $\underline{u}(k)$ is equivalent to a MAS with state $z(k)$ and input $v(k)$ executing Protocol 1 and estimating $\overline{v}(k)$, for which it holds the proof for Protocol 1. □

In the next corollaries we make explicit the convergence time in the case of optimal initialization and the steady-state error in the case of constant reference signals.

Corollary 7.3.4 *The convergence time for Theorem 7.3.3 in the case of an optimal initialization $x_i(0) = u_i(0)$ satisfy the following condition for $k \geq T_c > 0$*

$$T_c \leq \delta_{\mathcal{G}}. \tag{7.19}$$

Corollary 7.3.5 *The estimation errors for Theorem 7.3.3 in the case of constant reference signals, i.e., $\Pi = 0$, satisfy the following strict condition for $k \geq T_c \geq 0$*

$$e(k) = \alpha\delta_{\mathcal{G}}. \tag{7.20}$$

Remark 7.3.2 In the case of constant inputs, i.e., $\Pi = 0$, the choice of the parameter α must satisfy $\alpha > 0$. The case of $\alpha = 0$ with optimal initialization $x_i(0) = u_i(0)$ corresponds to the popular protocols in eq. (7.9)-(7.10) which have been shown to be biased and unsuitable to solve a tracking problem at the beginning of this section. In fact, the error $e(k) = 0$ and the convergence time $T_c \leq \delta_{\mathcal{G}}$ are derived as a special case. We point out that for $\alpha = 0$ the results in Theorem 7.3.3, which assume an arbitrary initialization, do not hold.

From Theorem 7.3.3, it follows that to minimize the absolute estimation error one needs to choose the smallest α satisfying the design condition in eq. (7.16). On the other hand, α determines the convergence time T_c as explained to Remark (7.3.2), with smaller values of α giving a greater convergence time. Thus, the value of α trades-off estimation error and convergence time. It follows that a pragmatic design criterion for the choice of α is to first fix the desired steady-state error and then choose the largest α which allows to satisfy the error performance constraint while minimizing the convergence time.

Exact dynamic min/max-consensus (EDMAC/EDMIC) protocols

In this section we consider networks of agents with vector state $x_i = [x_i^0, x_i^1, \ldots, x_i^m]^\intercal \in \mathbb{R}^m$ where $m \in \mathbb{N}$ is an upper bound on the diameter of the underlying communication network, i.e., $m \geq \delta_{\mathcal{G}}$. The strategy proposed in this section suggests to replicate the initialization step in eq. (7.8) of the conventional protocol at each instant of time in the first element x_i^0 of the state vector and then cascade the conventional protocol in eq. (7.6)-(7.7) over the remaining state variables: the estimate of each agent is the last state x_i^m. The proposed local interaction rules are formalized next

$$x_i^0(k) = u_i(k)$$

$$x_i^\ell(k) = \max_{j \in \mathcal{N}_i^\circ} \left\{ x_j^{\ell-1}(k-1) \right\}, \quad \ell = 1, \ldots, m, \tag{7.21}$$

$$x_i^\ell(k) = \min_{j \in \mathcal{N}_i^\circ} \left\{ x_j^{\ell-1}(k-1) \right\}, \quad \ell = 1, \ldots, m, \tag{7.22}$$

and the output is the last state variable,

$$y_i(k) = x_i^m(k),$$

Protocol 3: Exact Dynamic Max-Consensus (EDMAC)

Initialization: $x_i(0) \in \mathbb{R}$ for $i \in V$.
Init. for opt. conv. time: $x_i(0) = u_i(0)$ for $i \in V$.
Input: Network's diameter upper bound $m \in \mathbb{N}$.
Output: $y_i(k) = x_i^m(k) \in \mathbb{R}$ for $i \in V$.

for $k = 0, 1, 2, \ldots$ **each node** i **do**
  Gather $x_j(k)$ from each neighbor $j \in \mathcal{N}_i$
  Update the first current state according to
  $x_i^0(k) = u_i(k)$
  for $\ell = 1, \ldots, m$ **do**
    Update the current state according to
    $x_i^\ell(k) = \max_{j \in \mathcal{N}_i^\circ} x_j^{\ell-1}(k-1)$

which represents the current estimated value.

In Protocols 3-4 we detail the Exact Dynamic Max-Consensus (EDMAC) and the Exact Dynamic min-Consensus (EDMIC) Protocols, respectively, which make use of the two local interaction rules described above. The two protocols are characterized in the next theorem.

Theorem 7.3.6 *Consider a MAS executing either Protocol 3 or Protocol 4 under Assumption 7.2.1 and consider a generic initial instant of time $k_0 \in \mathbb{N}$. If graph $\mathcal{G}$ is connected and if*

$$m \geq \delta_{\mathcal{G}}, \tag{7.23}$$

then there exists a convergence time T_c such that the tracking errors defined in eq. (7.5) *are bounded for $k \geq T_c > 0$ by*

$$e(k) \leq |\Delta u(k, m)| \leq m\Pi, \tag{7.24}$$

where[4] *$e(k)$ and $\Delta u(k)$ are given in* (7.5) *and* (7.4)*, and it holds*

$$T_c = m. \tag{7.25}$$

4: The error $e(k)$ and the change $\Delta u(k, \cdot)$ are to be understood as the error on the max value $\overline{e}(k)$ and the change of the max value $\overline{\Delta u}(k, \cdot)$ for Protocol 1 and the error on the min value $\underline{e}(k)$ and the change of the min value $\underline{\Delta u}(k, \cdot)$ for Protocol 2.

Proof. We first prove the theorem with Protocol 1 and then the proof of the theorem with Protocol 2 is derived as a special case by showing its equivalence with Protocol 1.

Protocol 4: Exact Dynamic max-Consensus (EDMIC)

Initialization: $x_i(0) \in \mathbb{R}$ for $i \in V$.
Init. for opt. conv. time: $x_i(0) = u_i(0)$ for $i \in V$.
Input: Network's diameter upper bound $m \in \mathbb{N}$.
Output: $y_i(k) = x_i^m(k) \in \mathbb{R}$ for $i \in V$.

```
for k = 0, 1, 2, ... each node i do
    Gather x_j(k) from each neighbor j ∈ N_i
    Update the first current state according to
    x_i^0(k) = u_i(k)
    for ℓ = 1, ..., m do
        Update the current state according to
        x_i^ℓ(k) = min_{j ∈ N_i°} x_j^{ℓ-1}(k - 1)
```

Thus, let us first consider Protocol 1. At time k_0, we define the set

$$\mathcal{V}_0 = \left\{ i \in V : x_i^0(k_0) = \max_{j \in V} x_j^0(k_0) \right\}.$$

Since by Protocol 3 it holds $x_i^0(k) = u_i(k)$, then

$$\mathcal{V}_0 = \left\{ i \in V : x_i^0(k_0) = \overline{u}(k_0) \right\}.$$

Let us now consider time the set $\mathcal{V}_1$ of one-hop neighbors of nodes in set $\mathcal{V}_0$. Formally,

$$\mathcal{V}_1 = \left\{ i \in \mathcal{V} : (i, j) \in E,\ j \in \mathcal{V}_0 \right\}.$$

The update rule (7.21) of the state x_i^1 for agents belonging to this set reduces to

$$x_i^1(k_0 + 1) = \overline{u}(k_0), \quad \forall i \in \mathcal{V}_1$$

because all agents $i \in \mathcal{V}_1$ have a neighbor $j \in \mathcal{V}_0$ with state value $x_j^0(k_0) = \overline{u}(k_0 - 1)$. By induction, for $\ell \geq 1$ define

$$\mathcal{V}_\ell = \left\{ i \in V : (i, j) \in E, j \in \bigcup_{s=0}^{\ell-1} \mathcal{V}_s \right\},$$

and for all agents in these sets the update rule (7.11) of the state x_i^ℓ reduces to

$$x_i^\ell(k_0 + \ell) = \overline{u}(k_0).$$

By noticing that $\mathcal{V}_m \equiv \mathcal{V}_{\delta_\mathcal{G}} \equiv V$, we infer that for all $i \in V$ and for any time $k \geq k_0 + m$, it holds

$$x_i^m(k) = \overline{u}(k-m), \tag{7.26}$$

which proves that the convergence time T_c is equal to the upper bound m as in eq. (7.25). Furthermore, by Assumption 7.2.1 it holds $\overline{u}(k) = \overline{u}(k-m) + \overline{\Delta u}(k,m)$ and exploiting (7.26) we conclude that for any $k \geq m$ the bound on the tracking error given in (7.24) is correct since

$$\begin{aligned}
\overline{e}(k) &= \max_{i \in V} |y_i(k) - \overline{u}(k)| \\
&= \max_{i \in V} |x_i^m(k) - \overline{u}(k)| \\
&\leq \left|\overline{\Delta u}(k,m)\right| \leq m\Pi,
\end{aligned}$$

where the last inequality is due to Assumption 7.2.1.

Let us now consider Protocol 2. Let $v(k) = -u(k)$ for which it holds $\overline{v}(k) = -\underline{u}(k)$ and rewrite the local interaction rule as

$$\begin{aligned}
x_i^0(k) &= u_i(k) = -v_i(k) \\
x_i^\ell(k) &= \min_{j \in \mathcal{N}_i^\circ} \left\{x_j^{\ell-1}(k-1)\right\}, \qquad & \ell = 1, \ldots, m, \\
&= -\max_{j \in \mathcal{N}_i^\circ} \left\{-x_j^{\ell-1}(k-1)\right\}, \qquad & \ell = 1, \ldots, m.
\end{aligned}$$

By letting $z_i(k) = -x_i(k)$, one concludes that the MAS with state $x(k)$ and input $u(k)$ executing Protocol 4 and estimating $\underline{u}(k)$ is equivalent to a MAS with state $z(k)$ and input $v(k)$ executing Protocol 3 and estimating $\overline{v}(k)$, for which holds the proof for Protocol 3. □

In the next corollary we make explicit the steady-state error in the case of constant reference signals.

Corollary 7.3.7 *The estimation errors for Theorem 7.3.6 in the case of constant reference signals are null,*

$$e(k) = 0, \qquad k > T_c = m. \tag{7.27}$$

Protocol 5: Dynamic Size-Estimation (DSE)

Input: Number of random numbers $p \in \mathbb{N}$.

Output: $\hat{n}_i(k) = \dfrac{-p}{\sum_{j=1}^{p} \log(y_{ij}(k))}$ for $i \in V(k)$.

```
for k = 0, 1, 2, ... each node i do
    if k = 0 or i ∈ V(k) \ V(k − 1) then
        u_iℓ(k) ~ U(0, 1) for ℓ = 1, ..., p
    for ℓ = 1, ..., p do
        Execute either Protocol 1 or Protocol 3 with inputs
          [u_1ℓ(k), ..., u_n(k)ℓ(k)]
```

7.4 Application to size estimation of dynamic networks

In this section we focus on the problem of computing the size of a network, i.e., the number of active nodes in it: we describe the interconnections at time $k \in \mathbb{N}$ among the $n(k)$ active agents with a time-varying graph $\mathcal{G}(k) = (V(k), E(k))$.

The complexity of this problem heavily depends on the assumptions on the network. Here we consider the framework of anonymous networks [238] wherein the agents cannot be identified within the network, thus guaranteeing security and privacy of the nodes but hindering their cooperation, and each node only knows its neighbors and has not information on the topology, or at most only a little information such as a bound on the network size. The size estimation problem in anonymous network counts a high number of interesting applications, e.g., maintenance purposes in ad-hoc wireless sensor networks [39], optimization of query access plans in internet-scale data networks [171], coordination of robotic agents [33], and so on. In the next subsection we describe and characterize our protocol for estimating the time-varying network's size.

Dynamic size-estimation protocol

Our strategy, which is formalized in Protocol 5 and characterized in Theorems-7.4.1-7.4.2, extends the one proposed in [225] to networks where the agents are free to join or leave at any time, thus resulting in a time-varying network's size $n(k)$ to be estimated by the agents. The strategy is based on statistical inference concepts and can be outlined in three main steps: generation, estimation and inference, described next.

1. (Generation) When a node i joins the network generates $p \in \mathbb{N}_+$ independent random numbers $u_i^\ell \in [0,1]$ from a uniform distribution, i.e., $u_i^\ell \sim U(0,1)$ with $\ell = 1,\dots,p$;
2. (Estimation) The $n(k)$ active nodes execute either Protocol 1 or Protocol 3, thus each node i computes p estimates $y_{i\ell}$ of the maximum value among each local set $[u_1^\ell,\dots,u_{n(k)}^\ell]$ with $\ell = 1,\dots,p$;
3. (Inference) Each node i infers the network size $\hat{n}_i(k)$ by maximum likelihood estimation from its own set of estimations $[y_i^1,\dots,y_i^p]$.

The number p of generations is a design parameter: the higher is the value of p, the better is the estimation at steady-state but the slower is the convergence rate.

Between any two changes in the network the estimation problem reduces to the max-consensus problem on static inputs. However, whenever an agent leaves or joins the network, the set of inputs change, and so do their maximum values. Protocol 1 and Protocol 3 guarantee the convergence of the agent's estimation to the new set of inputs thanks to their robustness to the initial condition. Intuitively, the rate at which agents leave or join the network is correlated to the change of the maximum values to be estimated and thus some critical scenarios may happen. Here, we just make the assumptions that our protocols can run a sufficiently high number of iterations such that an equilibrium is reached after each change of the network: the minimum dwell time Υ between two changes of the network ensured by Assumption 7.2.2 is required to be greater or equal than the convergence time $T^c(k_0)$ of the employed protocol.

Theorem 7.4.1 *Consider an OMAS executing Protocol 5 under Assumptions 7.2.2 and consider a generic initial time $k_0 \in \mathbb{N}$ at which the network changes, i.e., $\mathcal{G}(k_0-1) \neq \mathcal{G}(k_0)$.*

Adopting Protocol 1 under the conditions of Corollary 7.3.5 and if the dwell time is greater than the convergence time, i.e., $\Upsilon \geq T_c$, the expected value for $k \geq T_c$ is

$$\mathbb{E}\left[\hat{n}_i(k)\right] = \varepsilon^{p-1} e^{\varepsilon n p} (\varepsilon p)^p \Gamma(1-p, \varepsilon n p), \qquad (7.28)$$

where $\varepsilon = \delta_{\mathcal{G}}\alpha$ is the steady-state error due to Protocol 1 and $\Gamma(\cdot)$ denotes the upper incomplete gamma function[5].

5: The upper incomplete gamma function $\Gamma(a,x)$ is defined as follows $\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt$. There does not exist a closed form of this function, but it is usually implemented in programming platforms. For example, with MATLAB it can be computed with the command `igamma(a,x)`.

Theorem 7.4.2 *Consider an OMAS executing Protocol 5 under Assumptions 7.2.2 and consider a generic initial time $k_0 \in \mathbb{N}$ at which* *the network changes, i.e.,* $\mathcal{G}(k_0 - 1) \neq \mathcal{G}(k_0)$.

Adopting Protocol 3 under the conditions of Corollary 7.3.7 and if the dwell time is greater than the convergence time, i.e., $\Upsilon \geq T_c$, *the expected value for* $k \geq T_c$ *is*

$$\mathbb{E}\left[\hat{n}_i(k)\right] = \frac{np}{p-1} \tag{7.29}$$

Proof. The proof of the above theorems is given here.

Since by Assumption 7.2.2 the network remains unchanged for $k \in [k_0, k_0 + \Upsilon]$, then by Protocol 5 the reference signals and their maximum are constant in this interval of time and conditions of Corollaries 7.3.5-7.3.7 are met. Therefore, in the remaining of the proof we consider a generic time $k \geq T_c$ at which the estimation protocols have reached the steady-state and omit the time dependence (k).

Consider the samples of numbers $u_{1j}, \ldots, u_{nj}$ for any $j = 1, \ldots, p$. Each of these numbers is randomly generated with probability distribution function $P(a) = a$ for $a \in [0, 1]$ and $P(a) = 0$ otherwise. The maximum value of the sample

$$\overline{u}_j = \max_{j \in V} u_{ij}, \quad \forall j = 1, \ldots, p.$$

is the the n-th order statistics of the sample. Consider now the sample obtained by the n-th order statistics of each random number generated by the agents, i.e.,

$$\widetilde{u} = \{\overline{u}_1, \ldots, \overline{u}_p\}.$$

All variables in the sample are i.i.d. random variables with probability density function $p_n(a) = nP^{n-1}(a)$ depending on the parameter n. Thus, the likelihood function $\mathcal{L}(n|\tilde{u})$ can be computed as the product of the probability density functions,

$$\mathcal{L}(n|\tilde{u}) = \prod_{j=1}^{p} p_n(\overline{u}_j) = n^p \prod_{j=1}^{p} \overline{u}_j^{\,n-1}.$$

In practice, it is often convenient to work with the natural logarithm of the likelihood function, called the log-likelihood

$$\begin{aligned}\mathcal{L}^*(n|\tilde{u}) = \ln(\mathcal{L}(n|\tilde{u})) &= \ln\left(n^p \prod_{j=1}^{p} \overline{u}_j^{n-1}\right)\\ &= \ln(n^p) + \sum_{j=1}^{p} \ln\left(\overline{u}_j^{n-1}\right)\\ &= p\ln(n) + (n-1)\sum_{j=1}^{p} \ln \overline{u}_j.\end{aligned}$$

By computing the value $\hat{n}$ maximizing the log-likelihood function one obtains the best estimate of the size n of the network, which is given by

$$\hat{n} = \frac{-p}{\sum_{j=1}^{p} \ln(\overline{u}_j)}. \tag{7.30}$$

However, variables $\overline{u}_j$ are not known exactly at each node, and instead they know their estimate y_{ij}. Therefore, the best estimation $\hat{n}_i$ an agent can do is to implement the following

$$\hat{n}_i = \frac{-p}{\sum_{j=1}^{p} \ln(y_{ij})}, \quad \forall i \in V. \tag{7.31}$$

It is necessary to understand how the error arising from the use of $y_{ij} \neq \overline{u}_j$ affects the estimation of $\hat{n}$. We start our discussion taking into consideration the employment of Protocol 1 for which a non-null error is reached at steady-state. Then, as a special case for zero error, we derive the discussion for Protocol 3.

Discussion for Protocol 1

By Corollary 7.3.5 the steady-state error for Protocol 1 in the estimating of $\overline{u}_j$ is bounded by the following

$$e^j = \max_{i \in V} \left| y_{ij} - \overline{u}_j \right| \leq \delta_{\mathcal{G}} \cdot \alpha = \varepsilon. \tag{7.32}$$

A fundamental consideration, resulting from the constructing proof of Theorem 7.3.3 and Corollary 7.3.5, is that at steady-state the estimation y_{ij} of agent i of the quantity $\overline{u}_j$ is always an underestimation, i.e., $y_{ij} \leq \overline{u}_j$ for all $i \in V$. With this consideration in mind, it is easy to realize that the worst case is when at least one agent underestimates all variables $\overline{u}_j$ with maximum error $\varepsilon = \delta_{\mathcal{G}}\alpha$. Thus, we consider such a worst case scenario by assuming that

$$\exists i \in V: \quad y_{ij} = \overline{u}_j - \varepsilon, \quad \forall j = 1, \ldots, p. \tag{7.33}$$

Under condition (7.33) we obtain a lower bound $\hat{n}^*$ on the estimation $\hat{n}_i$ of each agent, which is obtained as follows

$$\begin{aligned}
\hat{n}_i &= \frac{-p}{\sum_{j=1}^{p} \ln\left(\overline{u}_j - \varepsilon\right)} \\
&= \frac{-p}{\sum_{j=1}^{p} \ln\left(\overline{u}_j(1 - \frac{\varepsilon}{\overline{u}_j})\right)} \\
&= \frac{-p}{\sum_{j=1}^{p} \left[\ln \overline{u}_j + \ln\left(1 - \frac{\varepsilon}{\overline{u}_j}\right)\right]} \\
&\geq \frac{-p}{\sum_{j=1}^{p} \left[\ln \overline{u}_j + \ln\left(1 - \varepsilon\right)\right]} \\
&\geq \frac{-p}{\sum_{\ell=1}^{p} \left(\ln \overline{u}_j - \varepsilon\right)} \\
&\geq \frac{p}{\sum_{j=1}^{p} (-\ln \overline{u}_j) + p\varepsilon} \\
&\geq \frac{1}{\frac{1}{p}\sum_{j=1}^{p} (-\ln \overline{u}_j) + \varepsilon} = \hat{n}^*
\end{aligned} \tag{7.34}$$

At the denominator of (7.34) we can recognize the term

$$\gamma = \frac{1}{p}\sum_{j=1}^{p} -\ln \overline{u}_j. \tag{7.35}$$

Now consider the following conceptual steps:

1. Variables $\overline{u}_j$ are beta random variables with shape parameters equal to $(n,1)$, since they are the n-th order statistics of a sample of n random numbers drawn from a continuous distribution;
2. Variables $-\ln \overline{u}_j$ are exponential random variables with rate n due to the equivalence to the beta distribution with parameters $(n,1)$;
3. Variable γ as in (7.35) is a gamma random variable with shape p and rate pn since they are the averaged sum of exponential functions.

Therefore, by means of the *law of the unconscious statistician*, we can calculate the expected value of $\hat{n}^*$ as follows

$$\mathbb{E}[\hat{n}^*] = \int_0^{\infty} f(x)g(x)dx, \tag{7.36}$$

where $f(x) = 1/(x + \varepsilon)$ is the relation between $\hat{n}^*$ and the gamma variable γ, while $g(x)$ is the probability density function of the gamma variable γ, i.e.,

$$g(x) = \frac{(np)^p}{(p-1)!} x^{p-1} e^{-npx}$$

Solution to (7.36) can be computed through several solver (we have used Wolfram|Alpha Pro engine) and it is giving by the following

$$\mathbb{E}[\hat{n}^*] = \varepsilon^{p-1} e^{\varepsilon np} (np)^p \Gamma(1-p, \varepsilon np),$$

where $\Gamma(p, x)$ is known as the upper incomplete gamma function. We point out that this expression holds for $n, p \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$ such that $n \geq 1$, $p > 1$ and $\varepsilon \geq 0$. This completes the first part of the proof.

By Corollary 7.3.7 the steady-state error in the estimating of $\overline{u}_j$ is bounded by the following

Discussion for Protocol 3

$$e^j = \max_{i \in V} \left| x^{ij} - \overline{u}_j \right| = 0 = \varepsilon.$$

Solution to (7.36) for $\varepsilon = 0$ is given by the following

$$\mathbb{E}[\hat{n}^*] = \frac{np}{p-1}.$$

We point out that this expression holds for $n, p \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$ such that $n \geq 1$, $p > 1$ and $\varepsilon \geq 0$. This completes the second part of the proof. This result is coherent to the expected value provided in [225], thus proving (7.29) and confirming that (7.28) is a generalization for $\varepsilon \geq 0$. □

7.5 Simulations and discussion

To illustrate the performance of the proposed protocols, simulation results are given in this section. First, we consider the DMAC and EDMAC Protocols simulating a worst-case scenario network with line topology. Second, we simulate the DMAC Protocol tracking a sinusoidal input for different choices of the design parameters, showing that convergence time and tracking error can be traded-off. Third, we apply these protocols in the context of distributed size estimation of open networks considering the case of scale-free networks with approximately fixed diameter. Without loss of generality in this section we limit the simulations to the dynamic max-consensus protocols; dual simulations for the dynamical min-consensus problem are omitted.

Example 1: comparison of DMAC and EDMAC Protocols

We simulate a network of $n = 10$ agents with line topology. The choice of the line topology is instrumental to run simulations in the worst case scenario. In fact, for line graphs the information takes exactly $\delta_{\mathcal{G}} = n - 1 = 9$ steps to flow through the network, thus maximizing the error for a fixed number of agents.

Figures 7.2-7.3 show the evolutions of the output variables (dashed red lines) and of the maximum among the time-varying inputs (solid blue line) when Protocol 1 or Protocol 3 are run over the MAS, respectively. The agents are uniformly initialized in the interval $[0, 1.5]$ and inputs are set to be equal to -1 for all nodes but the 6-th one, which is initialized at $u_6(0) = 0.5$ to be the maximum. All inputs remain constant except for the 6-th component, the maximum, which is time-varying with respect to the following

$$u_6(k+1) = \begin{cases} u_6(k) & \text{if } k < 100 \\ u_6(k) - \Pi & \text{if } k \in [100, 150) \\ u_6(k) & \text{if } k \in [150, 200) \\ u_6(k) + \Pi & \text{if } k \in [200, 250) \\ u_6(k) & \text{if } k \geq 250 \end{cases}, \tag{7.37}$$

with initial condition $u_6(0) = 0$ and $\Pi = 0.02$ being the absolute change according to Assumption 7.2.1. The design for the DMAC Protocol and its characterization provided by Theorem 7.3.3 are given next:

- Input parameter $\alpha = 0.021$;

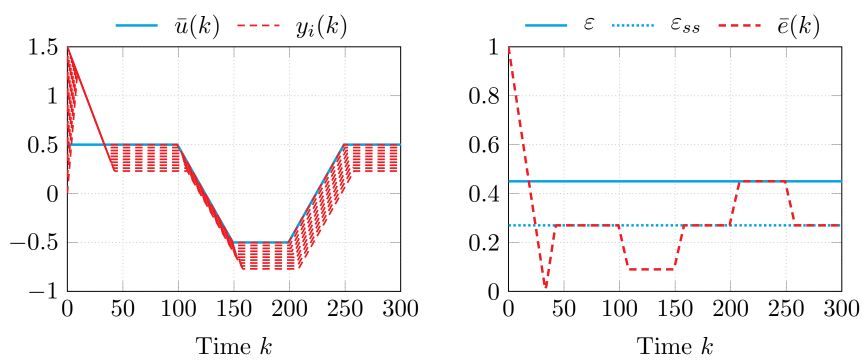

Figure 7.2: Evolution of a MAS evolving according to Protocol 1 in Example 1: ε denotes the bound on the tracking error as in eq. (7.24) and ε_{ss} denotes the bound on the steady-state error as in eq. (7.20).

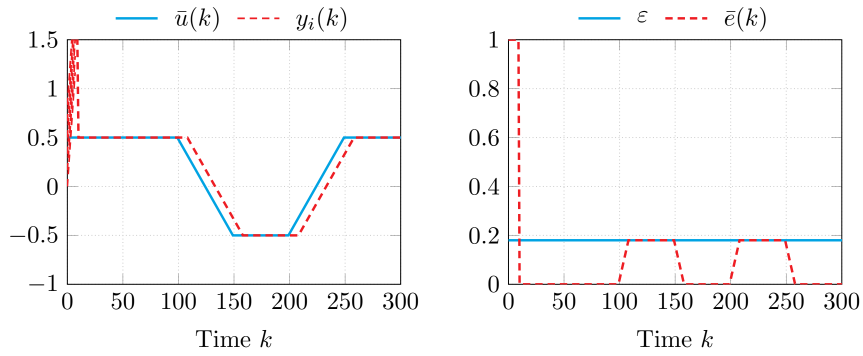

Figure 7.3: Evolution of a MAS evolving according to Protocol 3 in Example 1: ε denotes the bound on the tracking error as in eq. (7.17), while the steady-state error is zero.

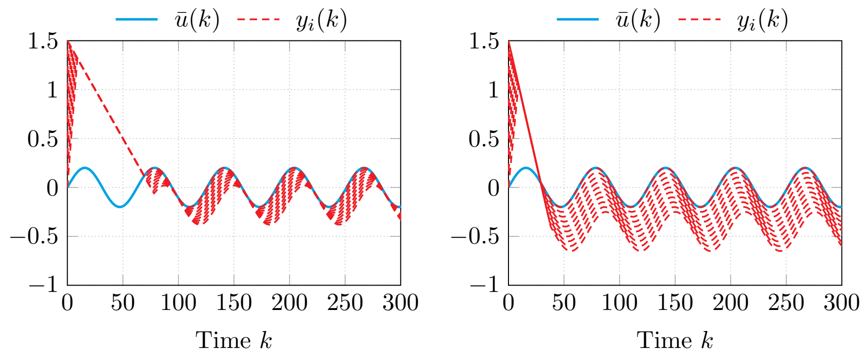

Figure 7.4: Evolution of a MAS evolving according to Protocol 1 in Example 2:.

- Convergence time $T_c = 32$;
- Bound on the tracking error $\varepsilon = 0.37$;
- Bound on the steady-state error $\varepsilon_{ss} = 0.19$.

The design for the EDMAC Protocol and its characterization provided by Theorem 7.3.6 are given next:

- Input parameter $m = 9$;
- Convergence time $T_c = 9$
- Bound on the tracking error $\varepsilon = 0.18$;
- Bound on the steady-state error $\varepsilon_{ss} = 0$.

These simulations show how the protocols steer the agents to track the time-varying maximum value $\overline{u}(k)$ among the reference signals, corroborating the characterization of the convergence times and the bound on the errors given in Theorems 7.3.3-7.3.6 and Corollaries 7.3.5-7.3.7.

Example 2: design trade-off for DMAC Protocol

As a second simulation we consider the same network and initialization of Section 7.5. The time-varying input $u_6(k)$ is a sinusoidal signal given by

$$u_6(k) = u(k) + 0.2 \sin\left(\frac{k}{10}\right),$$

with initial condition $u(0) = 0$, which is also the maximum signal to be tracked since all other inputs stay constant at -1. Notice that for this signal the change is bounded by $\Pi = 1.99$.

Fig. 7.4 show the evolution of the output variables (dashed red lines) and the maximum time-varying input (solid blue line). In particular, in the left plot the DMAC Protocol has been designed with $\alpha = 0.2$, while the right plot with $\alpha = 0.5$. We notice that the design of $\alpha = 0.2$ provides a greater convergence time $T_c = 70$ and a smaller tracking error $\bar{e} \leq 0.2$ compared to the design $\alpha = 0.5$ which gives a convergence time $T_c = 35$ and a tracking error $\bar{e}(k) \leq 0.4$. Thus, the design of α provides a trade-off between convergence time and tracking error.

Example 3: dynamic size estimation

We choose to run simulations of size estimation over scale-free networks [6, 9]. A scale-free network is a network whose degree distribution follows a power law, at least asymptotically. That is, the fraction of nodes in the network having k connections to other nodes goes for large values of k as $k^{-\gamma}$, where the parameter $\gamma \in \mathbb{R}$ typically is in the range $[2, 3]$. Such networks are known to be *ultrasmall*, as proved in [40], meaning that their diameter scales very slow with the dimension of the network, behaving as $d \sim \ln \ln N$.

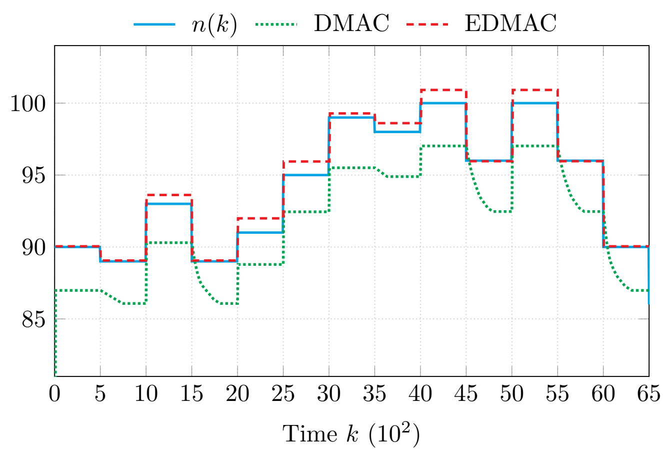

Figure 7.5: Dynamic Size Estimation of a network by means of Protocols 5.

We randomly generate a scale-free network by means of Barabási – Albert (BA) model proposed in [6]. This algorithm generates random scale-free networks using a preferential attachment mechanism given an initial small network, no necessarily scale-free. We use as initial network a line network of 5 nodes, and then we run the algorithm until a network of $n = 100$ nodes is generated. This network has a diameter of the order of the original small network, i.e., $d \approx 5$. In order to simulate nodes leaving and joining the network without losing the connectivity and the scale-free structure of the graph, we randomly deactivate or activate some of the last 25 nodes added to the network by the algorithm every $5 \cdot 10^2$ steps.

Fig. 7.5 shows the estimation of the size of a network by means of Protocol 5 which makes use of one of the dynamic max-consensus protocols proposed in Section 7.3, i.e., the DMAC protocol given in Protocol 1 and the EDMAC protocols given in Protocol 3; this strategy constitutes a generalization of the method proposed in [225] to open networks.

Average state observer 8

In this chapter we focus on the problem of estimating the average of unmeasured nodes in a network [144, 170, 197]. The development of state estimators is even a more critical task on large scale networks, where the dimension of the system is very high and thus a dense deployment of computational resources and sensing gateway devices is required [142, 188, 209]. Thus, in such networks the number of measured states p is assumed to be much smaller than the total number of states n, i.e., $n \gg p$. In particular, denoting with $k = n - p$ the number of unmeasured nodes, we consider linear and time-invariant networks where only a few nodes measurements are available,

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = \begin{bmatrix} \mathbb{0} & I_p \end{bmatrix} x(t)$$

and we consider the estimation of the average of the unmeasured part of the network, namely the quantity

$$z(t) = \begin{bmatrix} \frac{\mathbb{1}}{n}\mathbb{1}_k^\top & \mathbb{0}_p^\top \end{bmatrix} x(t).$$

[144] Luenberger (1971), 'An introduction to observers'.
[197] Sadamoto et al. (2017), 'Average State Observers for Large-Scale Network Systems'.
[170] Niazi et al. (2019), 'Average observability of large-scale network systems'.
[142] Liu et al. (2011), 'Controllability of complex networks'.
[209] Singh and Hahn (2005), 'State estimation for high-dimensional chemical processes'.
[188] Pilloni et al. (2013), 'Decentralized state estimation in connected systems'.

We first review the related literature in Section 8.1 and then in Section 8.2 we formally define the problem. The main contribution are developed in Section 8.3 and are threefold: 1) we propose two observer design procedures enabling the average estimation without being affected by the size of the system and thus suitable to be employed on large-scale networks; 2) we derive a necessary and sufficient condition for exact estimation; 3) due to the restrictive nature of the latter condition, we prove the boundedness of the asymptotic estimation error and devise an optimal choice of the design parameters achieving its minimization when such a condition is not met. Finally, in Section 8.4 simulations corroborating the theoretical results are provided.

8.1 Related literature

The estimation of the average state of a large-scale system has nothing to do with consensus on the average or with the other consensus problems we have previously considered in this thesis. Thus, in this section we review the literature related to this problem.

The fields of applications of this problem are several. For instance, having a real-time estimation of the average traffic intensity on some portion of a portion of a traffic network may help to promptly prevent congestion situations [42]. Similarly, it may be useful the estimation of the average proportion of people infected during an epidemic [152], or influenced by a leader opinion [139], in a specific geographic area through a network where people are interconnected on the basis of their common social activities. Other examples may include also the thermal comfort control of smart buildings through a sensor network [56] and the monitoring of power grids [56]. Such systems require tremendous amount of computational and sensing resources for monitoring purposes and the available computational resources often cannot handle the complexity of such systems. Such limitations make the full estimation of the network's state a challenging task. Nevertheless, for control and monitoring, the knowledge of an aggregated quantity of the state is often sufficient. For instance, in positive systems [191], the average of the state vector provides a suitable estimate for the state norm, which finds applications in feedback stabilization [127]. Thus we study the estimation of an average state for LTI systems in this paper.

We follow the approach of functional observers[48] and inspired by the rich literature based on clustered-oriented model reduction schemes [16], we address the problem from a lower-order projection of the original system. In particular, our goal is to estimate the average of unmeasured part of the state vector, which is aggregated to obtain a system with dimension equal to the number of measured states plus one. Such a projection technique has the benefit of enabling the design of a reduced-order average observer for large-scale systems, thus maintaining tractable and scale-free the complexity of the estimation task and overcoming strong conditions required by standard techniques [7, 71, 163]. However, the drawback is the emergence of a matched and unmatched unknown inputs that encode the information that has been left out through the projection. In the context of state-estimation on linear systems with unknown inputs, sliding mode techniques are often privileged to linear techniques because of their inherent robustness against certain parameter variations, and matched unknown inputs, cfr [76, 215] with [239]. Thus, These nonlinear observers are also well-suited to be combined with other robust control

[42] Coogan and Arcak (2015), 'A compartmental model for traffic networks and its dynamical behavior'.

[152] Mei et al. (2017), 'On the dynamics of deterministic epidemic propagation over networks'.

[139] Li et al. (2017), 'Modelling the public opinion transmission on social networks under opinion leaders'.

[56] Deng et al. (2010), 'Building thermal model reduction via aggregation of states'.

[56] Deng et al. (2010), 'Building thermal model reduction via aggregation of states'.

[48] Darouach (2000), 'Existence and design of functional observers for linear systems'.

[16] Antoulas (2005), *Approximation of large-scale dynamical systems*.

[163] Murdoch (1973), 'Observer design for a linear functional of the state vector'.

[7] Aldeen and Trinh (1999), 'Reduced-order linear functional observer for linear systems'.

[71] Fernando et al. (2010), 'Functional observability and the design of minimum order linear functional observers'.

[215] Spurgeon (2008), 'Sliding mode observers: a survey'.

[76] Floquet et al. (2007), 'On sliding mode observers for systems with unknown inputs'.

techniques to attenuate also unmatched perturbation affecting the system dynamics, see [36, 53]. This handles the issue of complexity in large-scale network systems and enables us to design an observer whose dimension does not scale with the dimension of the original system, hence the term "scale-free estimation," similar to which is introduced in [35]. The problem of estimating a function of the state vector arises in many applications.

[239] Yang and Wilde (1988), 'Observers for linear systems with unknown inputs'.

[53] De Loza et al. (2013), 'Unmatched uncertainties compensation based on high-order sliding mode observation'.

[36] Castaños and Fridman (2006), 'Analysis and design of integral sliding manifolds for systems with unmatched perturbations'.

[35] Casadei et al. (2018), 'Controllability of Large-Scale Networks: An Output Controllability Approach'.

8.2 Average state observation problem

Consider a linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{8.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^q$ is the input, $y(t) \in \mathbb{R}^p$ is the measured states at time $t \in \mathbb{R}_{\geq 0}$, and where $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times q}$, and $C = (O_{p \times k}, I_p) \in \mathbb{R}^{p \times n}$. Unless strictly necessary, in the reminder of the paper the dependence from t is dropped for economy.

Without loss of generality, the state vector is partitioned as $x = [x_u, y]^\top$, where $x_u \in \mathbb{R}^k$ is the vector of the unmeasured states and $y \in \mathbb{R}^p$ denotes the vector of the measured states. System (8.1) is thus rewritten as follows

$$\begin{bmatrix} \dot{x}_u \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}_{A} \begin{bmatrix} x_u \\ y \end{bmatrix} + \underbrace{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}_{B} u \tag{8.2}$$

In large-scale networks the number n of interconnected systems is very high and only a limited number p of gateway nodes is available for monitoring purposes, thus it usually holds that $p \ll n$. Because of that, usually the system is not observable, i.e., it is not possible to exactly estimate the whole unmeasured part of the system x_u with the available outputs y. However, in a certain number of applications, the problem of estimating a function of the state vector arises, without the need of estimating the entire state vector. In this work we focus on estimating the average of the unmeasured part of the state. Consider the linear time-invariant system (8.2) whose state is partitioned into unmeasured x_u and measured y. Design a state observer enabling the estimation of the following aggregated scalar function

$$z(t) = \frac{1}{k}\mathbb{1}_k^\top \cdot x_u(t), \tag{8.3}$$

which represents the averaged value of the unmeasured states of the system. This suggests one should seek functional observers with a reduced dimension with respect to a full-state observer. Following the approach proposed in [169, 196], we are going to project system (8.1) into a lower order system with state $z = [z, y]^\top$, in which the unknown average state $z(t)$ as in (8.3) is revealed. By letting

[196] Sadamoto et al. (2013), 'Low-dimensional functional observer design for linear systems via observer reduction approach'.
[169] Niazi et al. (2019), 'Scale-free estimation of the average state in large-scale systems'.

$$z = Px, \quad \text{with } P = \begin{bmatrix} \frac{1}{k}\mathbb{1}_k^\top & \mathbb{0}_p^\top \\ \mathbb{0}_{p\times k} & I_p \end{bmatrix} \in \mathbb{R}^{(p+1)\times n}, \tag{8.4}$$

one has the following reduced-order system

$$\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}}_{E} \begin{bmatrix} z \\ y \end{bmatrix} + \underbrace{\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}}_{G} u + \underbrace{\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}}_{F} \delta \tag{8.5}$$

where

$$E = \begin{bmatrix} \frac{1}{k}\mathbb{1}_k^\top A_{11}\mathbb{1}_k & \frac{1}{k}\mathbb{1}_k^\top A_{12} \\ A_{21}\mathbb{1}_k & A_{22} \end{bmatrix}, \; G = \begin{bmatrix} \frac{1}{k}\mathbb{1}_k^\top B_1 \\ B_2 \end{bmatrix}, \; F = \begin{bmatrix} \frac{1}{k}\mathbb{1}_k^\top A_{11} \\ A_{21} \end{bmatrix},$$

and $\delta(t) \in \mathbb{R}^k$ is an unknown input vector defined as

$$\delta(t) = \left(I_k - \frac{1}{k}\mathbb{1}_k \cdot \mathbb{1}_k^\top\right) x_u(t). \tag{8.6}$$

Notice that, as a result of lower order projection, it is not possible to reconstruct the space $x_u \in \mathbb{R}^k$ from $z \in \mathbb{R}$. In particular, the unknown input δ accounts for what is missed. It represents the element-wise deviation of each component of the unmeasured vector x_u to the average quantity $z(t)$ as in (8.3), thus its mean is zero at any time, i.e.,

$$\mathbb{1}_k^\top \cdot \delta(t) = 0, \qquad t \in \mathbb{R}_{\geq 0}. \tag{8.7}$$

The problem of estimating the time-varying average of the unmeasured states of a linear time-invariant system in (8.2) is equivalent to the problem of observing the whole state of its projected system in (8.5). If on one hand the reduction of dimensionality allows one to maintain tractable and scale-free the analysis, on the other it introduces the drawback of the unknown input $\delta(t)$ in (8.6), which encodes what is left out through the projection. In the remainder of the paper, we make the next standing assumptions.

Assumption 8.2.1 *The system* (8.2) *is stable and the inputs are bounded, namely, one of the following holds:*

- $\text{eig}(A) \subset C_{\leq 0}$ *and* $\int_0^\infty \|u(t)\| dt < \infty$,
- $\text{eig}(A) \subset C_{<0}$ *and* $\|u(t)\| < \infty$ *for all* $t \in \mathbb{R}_{\geq 0}$.

Assumption 8.2.2 *The pair* (E, CP^+) *of system* (8.5) *is observable.*

Given a matrix A, the matrix A^* denotes its pseudoinverse.

Assumption 8.2.3 *The actual value of* (8.3) *is bounded by a known function* $\bar{\delta}(t)$*, namely,* $\|\delta(t)\|_\infty \leq \bar{\delta}(t)$*, for all* $t \in \mathbb{R}_{\geq 0}$.

All the above assumptions are reasonable and widely accepted. Assumption 8.2.1 is reasonable in the estimation of aggregated quantities, such as the average, because it mainly concerns analysis and monitoring of existing networks rather than for their output-feedback stabilization; Assumption 8.2.2 is typical in the design of sliding mode observers, see [215]-[68]; Assumption 8.2.3 comes directly from Assumption 8.2.1, in fact the boundedness of the system (8.2) implies that the unknown input in (8.3) is bounded as well.

[215] Spurgeon (2008), 'Sliding mode observers: a survey'.

[68] Edwards and Spurgeon (1998), *Sliding mode control: theory and applications*.

In the next two subsections we present two observer designs enabling the average estimation of the average of unmeasured states of system in (8.2) whose dynamics is described by its projection system in (8.5), in spite of the unknown input vector $\delta(t)$.

8.3 Proposed observer designs

Linear observer design

We propose a linear observer taking the form

$$\begin{aligned} \dot{\hat{w}} &= E_1^* \hat{w} + E_2^* y + G^* u, \\ \hat{z} &= \hat{w} - Ly \end{aligned} \tag{8.8}$$

with

$$\begin{aligned} E_1^* &= E_{11} + LE_{21}, \qquad E_2^* = E_{12} + LE_{22} - E_1^* L, \\ G^* &= G_1 + LG_2, \qquad F^* = F_1 + LF_2 \\ L &= (\alpha \mathbb{1}_k^\top - F_1) F_2^+ \end{aligned} \tag{8.9}$$

where $\hat{z} \in \mathbb{R}$ represent the state estimate for $z \in \mathbb{R}$ and $L \in \mathbb{R}^{1\times p}$ is the only design parameter, which constitutes an open loop matrix gain. The average estimation error is given by

$$e_z = \hat{z} - z. \tag{8.10}$$

Sliding mode design

We propose an average observer taking the form

$$\begin{bmatrix} \dot{\hat{z}} \\ \dot{\hat{y}} \end{bmatrix} = \underbrace{\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}}_{E} \begin{bmatrix} \hat{z} \\ \hat{y} \end{bmatrix} + \underbrace{\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}}_{G} u + \underbrace{\begin{bmatrix} H_1 \\ H_2 \end{bmatrix}}_{H} e_y + \begin{bmatrix} L \\ -I \end{bmatrix} \nu \tag{8.11}$$

with

$$\begin{aligned} H_1 &= -E_{12}, \qquad & H_2 &= E_{22}^s - E_{22} \\ L &= (\alpha \mathbb{1}_k^\top - F_1) F_2^+ \end{aligned} \tag{8.12}$$

where $\hat{z} \in \mathbb{R}$ represent the state estimate for $z \in \mathbb{R}$, matrices $H_1 \in \mathbb{R}^{1\times p}$, $H_2 \in \mathbb{R}^{p\times p}$ and $L \in \mathbb{R}^{1\times p}$ are constant feedback gains, ν is a discontinuous vector defined as

$$\nu = \rho \cdot \operatorname{sign}(e_y), \quad \text{with} \quad \rho \in \mathbb{R}^+.$$

and the average and output estimation errors are

$$e_z = \hat{z} - z, \qquad e_y = \hat{y} - y. \tag{8.13}$$

The design parameters are the L, E_{22}^s and ρ. This observer design can be seen as an application of the Utkin observer [68, Chapter 6.2.1], with additional Luenberger-type gains and subjected to an unknown input vector $\delta(t)$ affecting the reference system model. Furthermore, we address also unmatched perturbation due to the presence of the unknown input in the dynamics of the unmeasured states, in contrast with the typical sliding mode observer formulations, cfr. [36, 53] with [68, 76, 215].

[53] De Loza et al. (2013), 'Unmatched uncertainties compensation based on high-order sliding mode observation'.
[36] Castaños and Fridman (2006), 'Analysis and design of integral sliding manifolds for systems with unmatched perturbations'.
[215] Spurgeon (2008), 'Sliding mode observers: a survey'.
[76] Floquet et al. (2007), 'On sliding mode observers for systems with unknown inputs'.
[68] Edwards and Spurgeon (1998), *Sliding mode control: theory and applications*.

Exact estimation

Theorem 8.3.1 *Consider the linear large-scale system in* (8.1) *with its lower order projection in* (8.5)*, and let Assumptions 8.2.1, 8.2.2, and 8.2.3 be in force. Consider one of the two observer designs:*

- *The linear observer* (8.8) *with design* (8.9)*;*

- *The sliding mode observer* (8.11) *with design* (8.12) *and* $\text{eig}\{E^s_{22}\} \in \mathbb{C}^p_{<0}$, *and* $\rho > \|E_{21}z\|_\infty + \|F_2\bar{\delta}\|_\infty$.

The average estimation error $e_z = \hat{z} - z$ *converges to zero exponentially with rate*

$$\beta(\alpha) = F_1(I - F_2^+ F_2)\mathbb{1}_k + \alpha \mathbb{1}_k^\top F_2^+ F_2 \mathbb{1}_k, \tag{8.14}$$

if and only if it holds that

$$\text{rank}\left\{\begin{bmatrix} \alpha \mathbb{1}_k^\top - F_1 \\ F_2 \end{bmatrix}\right\} = \text{rank}\{F_2\}, \tag{8.15}$$

with

$$\alpha < \frac{F_1(F_2^+ F_2 - T)\mathbb{1}_k}{\mathbb{1}_k^\top F_2^+ F_2 \mathbb{1}_k}. \tag{8.16}$$

then the average estimation error

Proof. The first step of the proof consists in showing that the average estimation error e_z for both the observer design has dynamics

$$\dot{e}_z = (E_{11} + LE_{21})e_z - (F_1 + LF_2)\delta.$$

For the linear design it can be derived by simple substitution, while for the sliding mode observer one proceeds as follows. By selecting the design matrices H_1 and H_2 accordingly with (8.12), the estimation errors in (8.13) have dynamics

$$\begin{aligned} \dot{e}_z &= E_{11}e_z - F_1\delta + L\nu \\ \dot{e}_y &= E_{21}e_z + E^s_{22}e_y - F_2\delta - \nu. \end{aligned}$$

Notice that the evolution of the average estimation error e_z is decoupled from that of the output estimation error e_y, but not the vice-versa. By means of the Popov-Belevitch-Hautus rank test [68, Proposition 3.3] and by Assumption 8.2.2, one derives that also the pair (E_{11}, E_{21}) is observable, thus that $E_{21} \neq 0$. Such condition is necessary as also discussed in [170].

[170] Niazi et al. (2019), 'Average observability of large-scale network systems'.

Since the unknown input $\delta(t)$ given in (8.6) is bounded by Assumption 8.2.1 and E^s_{22} can be arbitrarily designed such that $\text{eig}\{E^s_{22}\} \in \mathbb{C}^p_{<0}$, then e_y can be made bounded-input bounded-state stable. By considering the candidate Lyapunov function $V = \frac{1}{1}2e_y^\top \cdot e_y$ and by differentiating it along the trajectories of e_y, it results that $\dot{V} = e_y^\top \cdot \dot{e}_y < -\epsilon\sqrt{V} < 0$ with $\epsilon > 0$, if and only if $\rho \geq \|E_{21}z_1\|_\infty + \|F_2\bar{\delta}\|_\infty + \epsilon$. This implies that an ideal sliding motion $\dot{e}_y = e_y = 0_p$ is guaranteed to take place in finite time. During the sliding motion the error dynamics take the form

$$\dot{e}_z = E_{11}e_z - F_1\delta + L\nu_{eq} \qquad 0 = E_{21}e_z - F_2\delta - \nu_{eq},$$

and substituting for ν_{eq} yelds

$$\dot{e}_z = (E_{11} + LE_{21})e_z - (F_1 + LF_2)\delta. \tag{8.17}$$

The rest of the proof applies to both observer designs. By recalling property (8.6), it holds that

$$(F_1 + LF_2)\delta(t) = 0, \qquad \forall t \geq 0$$

if and only if $L \in \mathbb{R}^{p\times 1}$ is such that

$$F_1 + LF_2 = \alpha \mathbb{1}_k^\top, \tag{8.18}$$

with $\alpha \in \mathbb{R}$. By the Rouché–Capelli theorem, the system of linear equations (8.18), with L being the row vector of unknowns, is well-posed if and only if the condition (8.15) given in the statement of the theorem holds. In this case, the design of L, which is the same in the linear observer (see (8.9)) and in the sliding mode observer (see (8.12)), is the solution to the system of equations (8.18), and the error dynamics (8.17) reduces to $\dot{e}_z = (E_{11} + LE_{21})e_z$ where

$$\begin{aligned}(E_{11} + LE_{21}) &= F_1\mathbb{1}_k + LF_{21}\mathbb{1}_k \\ &= F_1\mathbb{1}_k + \left(\alpha\mathbb{1}_k^\top - F_1\right)F_2^+F_2\mathbb{1}_k \\ &= F_1(I - F_2^+F_2)\mathbb{1}_k + \alpha\mathbb{1}_k^\top F_2^+F_2\mathbb{1}_k \\ &= \beta(\alpha) < 0\end{aligned}$$

from which the last tuning constraint and the expression of the convergence rate are derived. □

Corollary 8.3.2 *Consider the linear large-scale system in* (8.1) *with its lower order projection in* (8.5)*, and let Assumptions 8.2.1, 8.2.2, and 8.2.3 be in force. Consider one of the observer design of Theorem 8.3.1 and condition* (8.15) *holding true. If and only if the next relation is satisfied*

$$\operatorname{rank}\left\{\left[\mathbb{1}_k^\top F_1\right]\right\} = 1, \tag{8.19}$$

then the convergence rate of the estimation error β given in (8.14) *can be chosen arbitrarily.*

Proof. The convergence rate of the estimation error $\beta(\alpha)$ is a function of α, thus it can be chosen arbitrarily if and only if α can be chosen arbitrarily. The parameter α can be chosen arbitrarily if and only if (8.19) holds true, since in this case α does not appear anymore in the rank condition (8.15), which reduces to

$$\operatorname{rank}\left\{\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}\right\} = \operatorname{rank}\left\{F_2\right\}.$$

□

It is very likely that a system as in (8.1) does not satisfy condition (8.15), thus preventing an observer designed according to Theorem 8.3.1 from achieving an exact estimation with zero error. In fact, for system (8.1) to satisfy (8.15), it is necessary that all unmeasured nodes are connected to at least one output node. This condition is however often unrealistic when the system size is large, because it not only requires a strategic deployment of sensors often impracticable, but it also requires a large number of measured nodes, while in most real cases only a limited number of measurement node is available. From a mathematical point of view, when this condition is not met the system of equations (8.18) is ill-posed and thus the effect of the unknown input $\delta(t)$ can not be nullified. In the next theorem, we show that even if (8.15) is not verified, an observer designed according to Theorem 8.3.1 provides an estimation of the average state (8.3) with bounded error.

Theorem 8.3.3 *Consider the linear large-scale system in* (8.1) *with its lower order projection in* (8.5), *and let Assumptions 8.2.1, 8.2.2, and 8.2.3 be in force. Consider one of the observer design of Theorem 8.3.1. Then, the average error e_z converges asymptotically within a boundary layer $\bar{e}_z$, i.e.,*

$$\lim_{t\to\infty} \|e_z(t)\| \leq \bar{e}_z. \tag{8.20}$$

Proof. Following the proof of Theorem 8.3.1, the average estimation error e_z is bounded-input bounded-state stable because of Assumption 8.2.1 and its dynamics is given by

$$\dot{e}_z = \beta(\alpha)e_z - (F_1 + LF_2)\delta.$$

However, if condition (8.15) is not satisfied, then the effect of the unknown input $\delta(t)$ can not be nullified anymore because the system of equations $F_1 + LF_2 = \alpha\mathbb{1}_k^{\intercal}$ is not well-posed. The average error trajectory $e_z(t)$ satisfies

$$|e_z(t)| \leq \left|e^{\beta(\alpha)\cdot t} e_z(0)\right| + \left|\int_0^{\top} e^{\beta(\alpha)\cdot(t-\tau)}(F_1 + LF_2)\delta(\tau)d\tau\right|.$$

Since α is strictly negative, then $\lim_{t\to\infty} |e^{\beta(\alpha)\cdot t}| = 0$. Moreover, we can derive

$$\lim_{t\to\infty} |e_z(t)| \leq \lim_{t\to\infty} \int_0^{\top} \left|e^{\beta(\alpha)(t-\tau)}(F_1 + LF_2)\delta(\tau)\right| \, d\tau$$

and by submultiplicativity of the norm

$$\lim_{t\to\infty} |e_z(t)| \leq \lim_{t\to\infty} \int_0^{\top} \left|e^{\beta(\alpha)(t-\tau)}\right| \cdot |(F_1 + LF_2)\delta(\tau)| \, d\tau.$$

Since the two functions argument of the above integral are non-negative by definition, it holds that (see [22, Sect. 23])

$$\lim_{t\to\infty} |e_z(t)| \leq \left[\lim_{t\to\infty} \int_0^{t} \left|e^{\beta(\alpha)(t-\tau)}\right| d\tau\right] \cdot \left[\max_{t\geq 0} \|(F_1 + LF_2)\delta(\tau)\|\right]$$

Finally, by a change of variable and solving the definite integral, one obtains the next upperbound

$$\lim_{t\to\infty} \int_0^{t} \left|e^{\beta(\alpha)(t-\tau)}\right| d\tau = \int_0^{\infty} \left|e^{\beta(\alpha)(\tau)}\right| d\tau = \frac{}{1}|\beta(\alpha)| \tag{8.21}$$

Considering (8.21), letting $\varphi = F_1 + LF_2 - \alpha \mathbb{1}_k^{\top}$ and invoking property (8.7), the following chain of inequalities takes place,

$$\begin{aligned}
\lim_{t\to\infty} |e_z(t)| &\leq \frac{}{1}|\beta(\alpha)| \max_{t\geq 0} \|(F_1 + LF_2)\delta(t)\|_2 \\
&\leq \frac{}{1}|\beta(\alpha)| \max_{t\geq 0} \left\|\left(\alpha \mathbb{1}_k^{\top} + \varphi\right) \delta(t)\right\|_2 \\
&\leq \frac{}{1}|\beta(\alpha)| \max_{t\geq 0} \left\|\varphi\delta(t)\right\|_2 \\
&\leq \frac{\|\varphi\|_2}{|\beta(\alpha)|} \max_{t\geq 0} \|\delta(t)\|_2
\end{aligned} \tag{8.22}$$

thus, recalling Assumption 8.2.3, the proof is complete. □

Moreover, the estimation provided by the observers is optimal in the sense that the effect of the unmatched part of the unknown input vector $\delta(t)$ is minimized, as proved in the following theorem.

Theorem 8.3.4 *Consider the linear large-scale system in* (8.1) *with its lower order projection in* (8.5)*, and let Assumptions 8.2.1, 8.2.2, and 8.2.3* *be in force. Consider one of the observer design of Theorem 8.3.1. Then, the bound on the average estimation error*

$$\bar{e}_z(\alpha) = \frac{\|F_1 + LF_2 - \alpha \mathbb{1}_k^\top\|_2}{|\beta(\alpha)|} \max_{t \geq 0} \|\delta(t)\|_2 \tag{8.23}$$

is minimized by the design of L.

Proof. We start by recalling the bound (8.22) obtained in the proof of Theorem 8.3.3,

$$\bar{e}_z = \frac{\|\varphi(L)\|_2}{|\beta(\alpha)|} \max_{t \geq 0} \|\delta(t)\|_2,$$

with

$$\varphi(L) = F_1 + LF_2 - \alpha \mathbb{1}_k^\top,$$

The bound on the error can be minimized by the choice of L by solving the following least square problem

$$L^* = \arg \min_{L \in \mathbb{R}^{1 \times p}} \|\varphi(L)\|_2 \tag{8.24}$$

According to the Projection theorem [145, p. 51], or similarly with [36, Proof of Proposition 2], the optimal solution L^* to (8.24) is given by

$$L^* = (\alpha \mathbb{1}_k^\top - F_1)F_2^+,$$

which corresponds to the design of L in both observers. □

From the constructing proof and result of the previous theorem and its corollary, it is straightforward to notice that not only the average estimation error is bounded by (8.20), but it is dependent on the parameter α, which is a free design parameter. Therefore, one can think of choosing the optimal α such that this error is minimized.

Corollary 8.3.5 *Consider the linear large-scale system in* (8.1) *with its lower order projection in* (8.5)*, and let Assumptions 8.2.1, 8.2.2, and 8.2.3 be in force. Consider one of the observer design of Theorem 8.3.1. Then, the choice*

$$\alpha = arg \min_{\alpha} \frac{\|\alpha p^\top - q^\top\|}{(\alpha p^\top - q^\top)\mathbb{1} + k\alpha} \quad with \quad \begin{aligned} p^\top &= \mathbb{1}^\top(F_2^+F_2 - I) \\ q^\top &= F_1(F_2^+F_2 - I) \end{aligned} \tag{8.25}$$

minimizes the average estimation error (8.20).

8.4 Simulations and discussion

Here numerical simulations illustrating the effectiveness of the proposed observer design procedures are discussed. Simulations were performed on the MATLAB®/Simulink environment with the Euler fixed-step solver and sampling time of 10^{-5} seconds. For the sake of clarity, in the following we denote with $A_d = \{a_{ij}\}$ the adjacency matrix representing graph $\mathcal{G}$, with $d_i = \sum_{i=1}^{n} a_{ij}$ the degree of each node and with $D = \text{diag}(d)$ the degree matrix, and with $L_p = D - A_d$ the Laplacian matrix of graph $\mathcal{G}$.

Example 1: Compartmental models are used to describe the flow of a substance between different parts (*compartments*) of a system. Each compartment is assumed to be an homogeneous entity, i.e., the distribution of the substance within it can be considered uniform. They are also often used to model population dynamics: in such a case, a compartment represent a class of individuals with the same property. Average estimation in such models is intriguing since one may not be interested in estimating the content of a substance in each compartment, but only the average content in a subset of them.

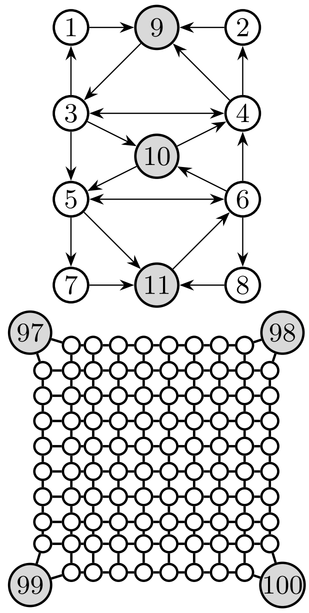

Figure 8.1: Graphs of Example 1 (above) and Example 2 (below)

Here, we consider a linear compartmental system [230] of $n = 11$ compartments, with $x_i(t) \in \mathbb{R}$ denoting the state of the i-th compartment and $x = [x_1^\top, x_2^\top, \dots, x_n^\top]^\top \in \mathbb{R}^n$ denoting the stacking of the states. Interconnections among the compartments are directed and according to Fig. 8.1: each directed edge (i, j) denotes the flow from compartment i to compartment j. Furthermore, white and grey nodes denote, respectively, the unmeasured and measured states, thus leading to the linear dynamical system in standard form of (8.1) where

[230] Walter and Contreras (2012), *Compartmental modeling with networks.*

$$A = -L_p, \qquad B = C^\top = \begin{bmatrix} 0_{3\times 8} & I_3 \end{bmatrix}$$

where L_p is the Laplacian matrix and the inputs are

$$u(t) = 10 \begin{bmatrix} \sin(t) & \sin(10t) & \sin(20t) \end{bmatrix}^\top.$$

By constructing the matrix A and its partitions A_{ij} with $i, j = \{1, 2\}$ it can be noticed that the necessary and sufficient conditions of Theorem 8.3.1 and Corollary 8.3.2 hold since $1^\top A_{11} = 1^\top A_{21} = 1^\top$, thus enabling exact average observation.

It can be noticed that both observers achieve asymptotic estimation with same decaying rate.

In order to effectively compare the proposed observer designs, we fix the convergence rate β of both observers at $\beta = -3$. The initialization of the system lies in the range $\mathbb{R}^n_{[0,3]}$. According to

It can be noticed that the choice of the parameter α trades-off convergence-time with estimation error.

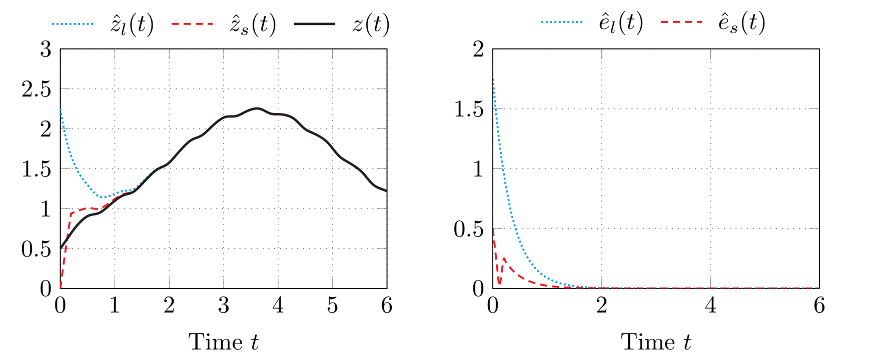

Figure 8.2: Simulation plots for the Example 1. On the left, the state trajectories of the system are plotted in black, while the estimations are given in blue for the linear observer and in red for the sliding mode observer. On the right, the absolute estimation errors are plotted.

Theorem 8.3.1, the linear observer and the sliding mode observer can be characterized by the following parameters

$$L^\top = -0.25 \cdot \mathbb{1}^\top, \qquad \rho = 5, \qquad E_{22}^s = -10 \cdot I.$$

Fig. 8.2 shows the trajectories of the observers and the average profile to be estimated, along with their estimation error. In particular:

- $\hat{z}_\ell(t)$ and $\hat{e}_\ell(t)$ denote the state estimation and the error estimation of the linear observer, respectively;
- $\hat{z}_s(t)$ and $\hat{e}_s(t)$ denote the state estimation and the error estimation of the sliding mode observer, respectively.

Example 2: Here, we consider a linear reaction-diffusion system [17, 98] of $n = 100$ substances, with $x_i(t) \in \mathbb{R}$ denoting the state of the i-th substance and $x = [x_1^\top, x_2^\top, \ldots, x_n^\top]^\top \in \mathbb{R}^n$ denoting the stacking of the states. Interconnections among the compartments are undirected and according to Fig. 8.1: each edge (i, j) or (j, i) denotes the transformation of substance i into j and vice versa. Furthermore, white and grey nodes denote, respectively, the unmeasured and measured states, thus leading to the linear dynamical system in standard form of (8.1) where

[17] Arcak (2011), 'Certifying spatially uniform behavior in reaction–diffusion PDE and compartmental ODE systems'.
[98] Hale (1997), 'Diffusive coupling, dissipation, and synchronization'.

$$A = -QL_p - \mathbb{R}, \qquad C^\top = \begin{bmatrix} 0_{4\times 96} & I_4 \end{bmatrix}$$

where L_p is the Laplacian matrix, $\mathbb{R}$, $Q \in \mathbb{R}^n$ are the diagonal matrices of diffusion coefficients and reaction rates, respectively, and the inputs are such that

$$\begin{aligned} u_1(t) &= \sin(0.1t) && \text{applied to nodes 97 and 98,} \\ u_2(t) &= \sin(t) && \text{applied to nodes 99 and 100,} \\ u_3(t) &= 1 && \text{applied to the other boundary nodes.} \end{aligned}$$

Having chosen $Q = I$ and $R = 0.2 \cdot I$, by constructing the matrix A and its partitions A_{ij} with $i, j = \{1, 2\}$ it can be noticed that the

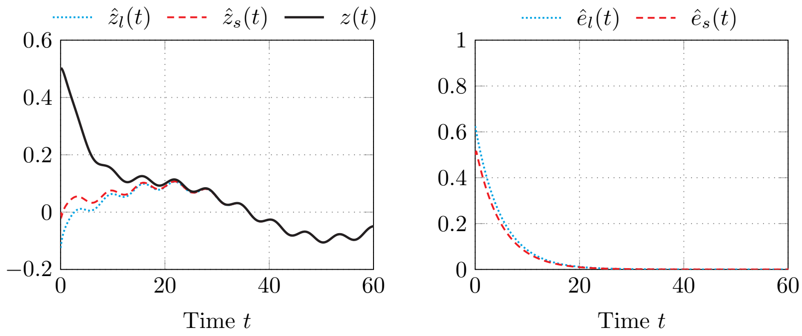

Figure 8.3: Simulation plots for the Example 2 with α minimizing the estimation error. On the left, the state trajectories of the system are plotted in black, while the estimations are given in blue for the linear observer and in red for the sliding mode observer. On the right, the absolute estimation errors are plotted.

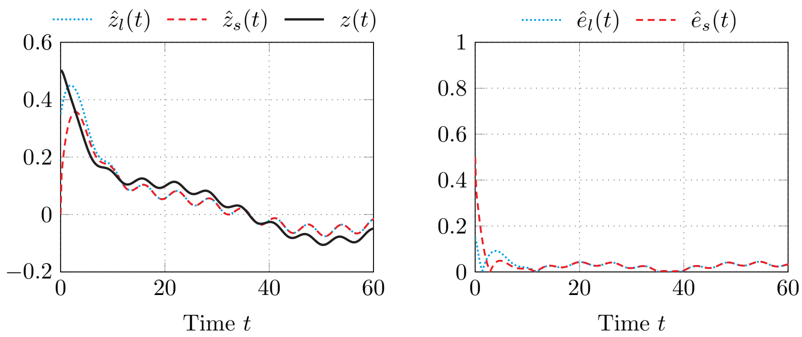

Figure 8.4: Simulation plots for the Example 2 with α guaranteeing a faster convergence rate. On the left, the state trajectories of the system are plotted in black, while the estimations are given in blue for the linear observer and in red for the sliding mode observer. On the right, the absolute estimation errors are plotted.

necessary and sufficient condition of Theorem 8.3.1 does not hold, thus preventing exact average observation.

As a first simulation, we choose $\alpha = -2.1 \cdot 10^{-3}$ which minimizes the estimation error for both observers and gives a decay rate equals to $\beta = 0.2$. The initialization of the system lies in the range $\mathbb{R}^n_{[0,3]}$. The observers are designed with

$$L^\top = -0.0104 \cdot \mathbb{1}^\top, \qquad \rho = 5, \qquad E^s_{22} = -10 \cdot I.$$

In contrast, we simulate also the case in which the decaying rate is not chosen to be the one minimizing the error, but is chosen in order to reduce the convergence time. For instance, we choose $\alpha = 4.2 \cdot 10^{-2}$ which gives a convergence rate $\beta = 0.4$. Figs. 8.3 and 8.4 shows the trajectories of the two observer and their estimation errors with the two design choices. In particular:

It can be noticed that the choice of the parameter α trades-off convergence-time with estimation error.

- $\hat{z}_\ell(t)$ and $\hat{e}_\ell(t)$ denote the state estimation and the error estimation of the linear observer, respectively;
- $\hat{z}_s(t)$ and $\hat{e}_s(t)$ denote the state estimation and the error estimation of the sliding mode observer, respectively.

Future directions

Concluding remarks 9

In this thesis, several research topics pertaining to the domain of multi-agent and large-scale dynamical systems have been addressed, and several improvements with respect to the state of the art have been provided.

In the first part of the thesis, we have considered nonlinear MASs which are characterized by the properties of order-preservation and subhomogeneity, which are a super class of linear MASs ruled by row-stochastic matrices. It is known that the Perron-Frobenius theory for row-stochastic matrices has immediate consequences for the iterative behavior of the matrix powers, which can be exploited to study the stability of trajectories of the associated linear system: consider a linear system

$$x(k+1) = Ax(k), \qquad x(k) \in \mathbb{R}^n$$

then, if A is row-stochastic and if A is primitive, then all trajectories of the MAS converge to an equilibrium point. By means of recent nonlinear extensions of Perron-Frobenius theory, we have generalized this result to nonlinear maps possessing the property of type-K order-preservation and plus-subhomogeneity: consider a nonlinear system

$$x(k+1) = f(x(k)), \qquad x(k) \in \mathbb{R}^n$$

then, if f is type-K order-preserving and plus-subhomogeneous and if f contains at least a fixed point, then all trajectories of the system converges to an equilibrium point. A similar result has been provided for nonlinear systems ruled by subhomogeneous maps acting on the positive orthant $\mathbb{R}^n_+$. These results, which are given in Chapter 3, improve the state-of-the-art since they identify classes of nonlinear systems for which global asymptotical stability can be established by only checking that there exists at least an equilibrium point. Moreover, the proof techniques do not rely on standard methods (e.g., Lyapunov theory) and thus they can be used as an alternative strategy to perform a stability analysis.

In subsequent Chapters 4-5 these key results have been exploited in the context of nonlinear MASs whose vector field is differentiable, both in discrete and continuous-time. We have provided sufficient conditions on the local interaction rule f_i of the agents in a network which ensure that the global map f falls in the above mentioned classes of nonlinear maps, establishing in this way the stability of their trajectories toward an equilibrium point. Such sufficient conditions concern the sign structure of the Jacobian matrix J_f of map f and constitutes a generalization of the well-known Kamke-condition to type-K order-preserving systems. Moreover, the problem of consensus has been solved by showing that if the consensus space contains only equilibrium points, the presence of a globally reachable node, which is a standard connectivity assumption on the network topology, is sufficient to solve the problem. This theory advances the state-of-the-art since it characterizes a general family of nonlinear MASs for which stability results and convergence results to the consensus state are available, thus constituting a novel tool to their analysis and design.

In the second part of the thesis three different problems related to dynamic estimation in large network have been considered.

In Chapter 6 the problem of inducing desynchronization in a network of diffusively coupled harmonic oscillators has been formalized and solved by means of a local mean-field feedback to the oscillators. The problem of desynchronization in harmonic oscillators is made more complicated than standard phase oscillators by the fact that each oscillator may oscillate at different amplitudes: our definition of desynchronization coincides with a null-average behavior of the oscillators' outputs, excluding trivial non-oscillatory behaviors. We have provided the first design of a local control action solving this problem, which exploits the presence of the Fiedler vector as dominant eigenvector associated to a couple of imaginary conjiugate eigenvalues to make the network oscillate with zero mean and thus achieving desynchronization in the sense described above. It has also been proven that, and this is quite intuitive, the employment of the proposed control action in networks of single integrator agents constitutes a novel protocol to solve the distributed estimation of the Fiedler vector. This protocol is shown to be faster and more accurate then other protocols in the literature, other than not requiring a specific initialization of the network. The only drawback of the proposed protocol is the requirement of the knowledge of the algebraic connectivity of the graph, which makes it inoperable in time-varying networks.

In Chapter 7 the problem of counting the number of active agents in an open network has been considered. The problem is complicated since the agents in the network do not have any precise knowledge about the network and their identity is assumed to be kept hidden in the network. An interesting strategy is to make each agent extract a random number out of a distribution, and then infer the number of agents who took part to the experiment from some function of the randomly generated numbers. In particular, we have formally described how this strategy can be implemented in a totally distributed way: each agent generates the number as soon as it joins the network then run a distributed algorithm to estimate the maximum among the generated numbers and infer the number of active agents by Maximum Likelihood estimation. However, the actual literature lacks of a dynamic distributed algorithm to solve the consensus problem on the max time-varying value. Thus, we have proposed the two protocols solving for the first time the dynamic consensus problem on the max value. The protocols converge in finite time and guarantee an apriori bound on the tracking error. These protocols are employed and characrerized to dynamically estimate the size of a network, which was an unsolved problem of the literature.

In Chapter 8 we have considered the problem of monitoring the average state of a large-scale network when only a few measurements are available to a centralized observer. This problem is totally different from the consensus problem considered in previous chapter. The strategy employed to solve the problem considers a projection of the original system to a lower order system and designing observers for the reduced system; such approach is especially effective when dealing with large networks since it reduces the complexity of the problem. We have proposed two different observer designs, a linear design and a sliding mode design: in particular, the dimension of these observer do not scale with the dimension of the network. Necessary and sufficient conditions for the existence of these observers are provided, which are however very restrictive. Thus, we have also characterized the performance of this observers in the case of these conditions are not met, and we have shown that a bounded error can always be achieved and traded-off with convergence time.

9.1 Open problems

The research problems addressed and the results proposed in this thesis have left many open problems and paved the way for novel research directions.

First part

The stability and convergence results presented in Chapter 3 may be possibly extended to more complicated scenarions. We make here a brief list of possible extensions:

- Consider maps which are not differentiable: for instance, the author believes that the results may be extended to maps which are sub-differentiable;
- Consider time-varying maps: switching, time-dependent, state-dependent and so on;
- Consider nonlinear spaces and time-variant cones;
- Consider non-autonomous agents.

These extensions would be then exploited in the context of MASs, thus allowing a straighforward generalization of results in Chapters 4-5 to the scenarios described above. We remark that studying the consensus problem in nonlinear MASs under the above suggested examples is a very difficult task: in the current literature no general few results of this kind are provided and most of it consider specific protocols instead of trying to provide general class of systems. Another interesting problem is the one of considering open MASs, where the agents are allowed to join or leave the network.

Moreover, it could be of great interest to extend the results for the specific applications considered in the latter chapters as suggested next:

- The protocol proposed in Chapter 4 to solve the consensus problem on the max value in discrete-time could be generalized to the continuous case. Another interesting application could be trying to generalize the protocol to estimate the maximum of time-varying reference signals given as inputs to the agents, which is an open problem in discrete-time MASs;
- The synchronization results provided in Chapter 5 for generalized oscillators with directed coupling do not provide a sharp characterization of the coupling functions ensuring global synchronization conditions, thus this problem remains open. We remark that it has been highlighted that such a design exists, but it is not provided for arbitrary network topologies.

Second part

The main drawback of the local control actions proposed in Chapter 6 to solve the desynchronization problem in networks of diffusively coupled identical harmonic oscillators and to distributedly estimate the Fiedler vector in networks of single integrators is the assumption on the a-priori knowledge of algebraic connectivity. Thus, the most important extension of the proposed protocol would be to design a coupled dynamic estimator of the algebraic connectivity. This extension would enable the desynchronization in networks with time-varying and open networks. Other possible extensions include directed topologies and heterogeneity of the oscillators.

The two protocols proposed in Chapter 7 to solve the dynamic consensus problem on the min/max value could be extended to deal with several challenging assumptions on the network and the communication channels, such as time-delays, noise, packed-losses, outliers and many others. However, even without considering these complicated scenarios, one could think to extend the protocols by improving their performance. For instance, the author believes that the DMAC/DMIC protocols could be improved by some adaptive design of the design parameter, while the EDMAC/EDMIC protocols could be improved by a dynamic estimation of the network's diameter.

The two observer designs proposed in Chapter 8 to enable the estimation of the average of the unmeasured states in a large-scale networks require the knowledge of the dynamics of the networks. Therefore, a useful extension would dealing with uncertainties in the model of the network or measurement errors and disturbances.

9.2 List of papers by the author

International journals:

(J5) D. Deplano, M. Franceschelli, A. Giua, "Dynamic Min and Max Consensus and SizeEstimation of Anonymous Multi-Agent Networks", Transaction on Automatic Control (TAC) under review in 2020. Preprint available at arXiv:2009.03858.

(J4) D. Deplano, M. Franceschelli, A. Giua, L. Scardovi, "Desynchronization of Distributed Fiedler Vector Estimation with

Application to Desynchronization of Harmonic Oscillator Networks", IEEE Control Systems Letters (L-CSS), 2020.

(J3) D. Deplano, M. Franceschelli, S. Ware, R. Su, A. Giua, "A Discrete Event Formulation for Multi-Robot Collision Avoidance on Pre-planned Trajectories", IEEE Access, 2020.

(J2) D. Deplano, M. Franceschelli, A. Giua, "A Nonlinear Perron-Frobenius Approach for Stability and Consensus of Discrete-Time MASs", Automatica, 2020.

(J1) M. U. B. Niazi, D. Deplano, C. Canudas-de-Wit, and A. Y. Kibangou, "Scale-Free Estimation of the Average State in Large-Scale Systems", IEEE Control Systems Letters (L-CSS), 2019.

International conference proceedings:

(C5) D. Deplano, M. Franceschelli, A. Giua, L. Scardovi, "Desynchronization of Networks of Coupled Harmonic Oscillators via Distributed Fiedler Vector Estimation", 59th IEEE Conference on Decision and Control (CDC), 2020.

(C4) M. U. B. Niazi, D. Deplano, C. Canudas-de-Wit, and A. Y. Kibangou, "Scale-Free Estimation of the Average State in Large-Scale Systems", 58th IEEE Conference on Decision and Control (CDC), 2019.

(C3) D. Deplano, M. Franceschelli, A. Giua, "Discrete-Time Dynamic Consensus on the Max Value", 15th European Workshop on Advanced Control and Diagnosis (ACD), 2019.

(C2) D. Deplano, M. Franceschelli, A. Giua, "Lyapunov-free Analysis for Consensus of Nonlinear Discrete-time MASs", 57th IEEE Conference on Decision and Control (CDC), 2018.

(C1) D. Deplano, S. Ware, R. Su and A. Giua, "A Heuristic Algorithm to Optimize Execution Time of Multi-Robot Path", 13th IEEE International Conference on Control and Automation (ICCA), 2017.

Abstracts at national congresses:

(N2) D. Deplano, M. Franceschelli, A. Giua, "Stability and Consensus Analysis for a Class of Nonlinear Discrete-Time MAS", Automatica.it, Ancona, Italy, 2019. Winner of Best presentation awarded by SIDRA.

(N1) D. Deplano, M. Franceschelli, A. Giua, "Discrete-Time Dynamic Consensus on the Max Value", Automatica.it, Cagliari, Italy, 2020.

Appendix A

The sets $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}$ denote respectively the set of real, natural and integer numbers; also $\mathbb{R}_+$ and $\mathbb{Z}_+$ are the set of non-negative real numbers and non-negative integer numbers. The set $\mathbb{C}$ denotes the set of complex numbers. We let $\mathbb{1}_n \in \mathbb{R}^n$ (respectively $\mathbb{0}_n \in \mathbb{R}^n$) be the column vector of dimension n with all entries equal to 1 (respectively 0); the subscript is omitted if clear from the context.

Most of the contents in these appendices takes inspiration from [32].

[32] Bullo (2018), *Lectures on Network Systems*.

A.1 Periodic points of a dynamical system

Discrete-time

Consider a discrete-time dynamical system

$$x(k+1) = f(x(k).$$

The state $x(k)$ of the system at time $k \in \mathbb{N}$ given an initial condition $x(0) = x_0 \in X$ is given by $f^k(x_0)$. The *trajectory* of system (2.1) starting at x_0 is given by $\mathcal{T}(x, f) = \{f^k(x_0) : k \in \mathbb{Z}\}$. If the map $f(\cdot)$ is clear from the context, we simply write $\mathcal{T}(x_0)$. An trajectory is said to be *periodic* of period $p \in \mathbb{Z}_+$ if for all $x_p \in \mathcal{T}(x_0, f)$ it holds $f^p(x_p) = x_p$, and x_p is called a *periodic point*. A periodic point of period $p = 1$ is a *fixed* or *equilibrium* point x_e and $f(x_e) = x_e$. We denote $F(f) = \{x \in X : f(x) = x\}$ the set of all equilibrium points of map $f(\cdot)$. An equilibrium point x_e is said to be Liapunov stable if for every $\varepsilon > 0$ there is $\delta > 0$ such that $||x_0 - x_e|| < \delta$ implies $||f^k(x_0) - x_e|| < \varepsilon$ for any $x_0 \in \mathbb{R}$ and $k \in \mathbb{Z}_+$.

Continuous-time

Consider a continuous-time dynamical system

$$\dot{x}(t) = f(x(t)).$$

The state $x(t)$ of the system at time t given an initial condition $x(0) = x_0$ is given by $\varphi(t, x_0)$, where $\varphi(t, x_0)$ denotes the solution to the initial value problem with initial condition x_0. The *trajectory* of system (2.2) starting at x_0 is given by $\mathcal{T}(x_0, f) = \{\varphi(t, x_0) : t \in \mathbb{R}\}$. If the vector field $f(\cdot)$ is clear from the context, we simply write $\mathcal{T}(x_0)$. An trajectory is said to be *periodic* of period $T \in \mathbb{R}_+$ if for all $x \in \mathcal{T}(x_0, f)$ it holds $\varphi(t+T, x) = x$. A periodic trajectory of period $T = 0$ contains only one point x_e, which is called an *equilibrium point* and $f(x_e) = 0$. We denote $F(f) = \{x \in X : f(x) = 0\}$ the set of all equilibrium points of vector field $f(\cdot)$. An equilibrium point x_e is said to be Lyapunov stable if for every $\varepsilon > 0$ there is $\delta > 0$ such that $||x_0 - x_e|| < \delta$ implies $||\varphi(t, x_0) - x_e|| < \varepsilon$ for any $x_0 \in \mathbb{R}$ and $t \in \mathbb{R}_+$.

A.2 Elements of matrix theory

We let I_n denote the n-dimensional *identity matrix*, i.e., a square matrix of order n with elements in the diagonal equal to 1 and all others equal to 0. Let $A = \{a_{ij}\} \in \mathbb{R}^{n\times n}$ denote a square $n \times n$ matrix with real entries $a_{ij} \in \mathbb{R}$ with $i, j \in \{1, \dots, n\}$. The matrix A is *symmetric* if $A^Tr = A$.

For a matrix $A \in \mathbb{R}^{n\times n}$, $\lambda \in \mathbb{C}$ is an *eigenvalue* and $v \in \mathbb{C}^n$ is an *eigenvector* if it holds $Av = \lambda v$. Given an eigenvalue λ of A: its *algebraic multiplicity* is the multiplicity of λ as a root of the characteristic polynomial $p(A) = \det(A - \lambda I)$; its *geometric multiplicity* is the number of linearly-independent eigenvectors associated to it. An eigenvalue is *simple* if it has algebraic and geometric multiplicity equal precisely to 1, and it is *semisimple* if the algebraic and geometric multiplicity are equal. The *spectrum* of A, denoted as $\sigma(A)$, is the set of eigenvalues of A. The *spectral radius* of A, denoted as $\rho(A)$, is the maximum norm of the eigenvalues of A, that is, $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$.

A.3 Elements of graph theory

By using notation from algebraic graph theory, we model the pattern of interactions among the agents in a network with a *graph* $\mathcal{G}(V, E)$, which consists of a set $V = \{1, \dots, n\}$ of *nodes* representing the agents and a set $E = V \times V$ of pairs of nodes (j, i) with $i, j \in V$, called *edges*. The terms node and agent are often used as synonyms.

If not otherwise stated, a graph is considered to be a *directed graph*, i.e., each ordered pair $(j, i) \in E$ denotes a *directed edge* from node j to node i.If $(j, i) \in E$, then node j is said to be a neighbor of agent i; the set of neighbors of the i-th node is denoted as $\mathcal{N}_i = \{j \in V : (j, i) \in E\}$. A self-loop is an edge from a node to itself.

A *directed path* between two nodes j and i is a finite sequence of m edges $(j_k, i_k) \in E$ that joins node j to node i, i.e., $j_1 = j$, $i_m = i$ and $i_k = j_{k+1}$ for $k = 1, \dots, m - 1$. A node i is said to be *reachable* from node j if there exists a directed path from node j to node i. A node i is said to be *globally reachable* if there exists a directed path from any node $j \in V$ to node i.

An edge $(j, i) \in E$ is usually to be intended as directed, but if also $(i, j) \in E$, then it is an *undirected edge*. If all edges in E are undirected, then it is an *undirected graph*, otherwise if at least one edge is directed, then the graph is said to be directed. Self-loops are not allowed in undirected graphs. Similar definitions apply to paths. An undirected graph is said to be *connected* if for any pair of nodes $i, j \in V$ there exists a path between them, otherwise it is said to be *disconnected*. For directed graphs, one can distinguish between *strong connectivity*, if there exists a directed path between any pair of nodes $i, j \in V$, and *weak connectivity*, if the undirected version of the graph is connected. The presence of a globally reachable node makes the graph a weakly connected graph.

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